1 Mass Conservation 14.12.2023

Let

$$\begin{aligned}\Phi_t: &\quad \Omega_0 \to \Omega_t, \quad \mathbf{x}_0 \mapsto \mathbf{x} \ = \mathbf{p}(t, t_0, \mathbf{x}_0),\\ \Phi_{-t}: &\quad \Omega_t \to \Omega_0, \quad \mathbf{x} \ \mapsto \mathbf{x}_0 = \mathbf{q}(t, t_0, \mathbf{x}) = \mathbf{p}(t_0, t, \mathbf{x}),\end{aligned}$$

be the flow of the vector field $\mathbf{V}$ with single field lines characterized by $\mathbf{p}(t, t_0, \mathbf{x}_0)$

$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{V}(t, \mathbf{p}(t)) = \mathbf{V} \circ \mathbf{p}, \qquad \mathbf{p}(t_0, t_0, \mathbf{x}_0) = \mathbf{x}_0, \tag{1}$$

i.e., for each starting point $\mathbf{x}_0$ is $\mathbf{p}$ the solution of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{V}(t, \mathbf{x}). \tag{2}$$

Ω_t a volume comoving with the flow and J the Jacobian of the transformation,

$$\Omega_t = \Phi_t(\Omega_0), \quad J(t, t_0, \mathbf{x}_0) = \frac{\partial \left(p^1, p^2, \cdots, p^n\right)}{\partial \left(x_0^1, x_0^2, \cdots, x_0^n\right)} = \det\left(\frac{\partial p^i}{\partial x_0^k}\right). \tag{3}$$

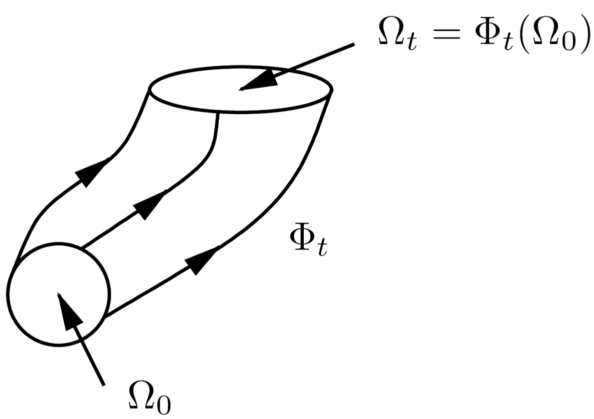

Figure 1: Integral lines of a vector field.

The flow $x^i = p^i(t, t_0, \mathbf{x}_0)$ induces a variable transformation in the differential volume element (hodge dual of 1),

$$\begin{aligned} *1 &= \frac{1}{n!}\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_n}\, \varepsilon_{i_1 i_2 \ldots i_n} \sqrt{g(\mathbf{x})} \\ &= \frac{1}{n!}\left(\frac{\partial p^{i_1}}{\partial x_0^{j_1}}\mathrm{d}x_0^{j_1}\right) \wedge \left(\frac{\partial p^{i_2}}{\partial x_0^{j_2}}\mathrm{d}x_0^{j_2}\right) \wedge \cdots \wedge \left(\frac{\partial p^{i_n}}{\partial x_0^{j_n}}\mathrm{d}x_0^{j_n}\right) \varepsilon_{i_1 i_2 \ldots i_n} \sqrt{g \circ \mathbf{p}} \\ &= \frac{1}{n!}\frac{\partial p^{i_1}}{\partial x_0^{j_1}}\frac{\partial p^{i_2}}{\partial x_0^{j_2}} \cdots \frac{\partial p^{i_n}}{\partial x_0^{j_n}}\, \varepsilon_{i_1 i_2 \ldots i_n} \mathrm{d}x_0^{j_1} \wedge \mathrm{d}x_0^{j_2} \wedge \cdots \wedge \mathrm{d}x_0^{j_n} \sqrt{g \circ \mathbf{p}} \\ &= \frac{1}{n!}J(t, t_0, \mathbf{x}_0)\, \varepsilon_{j_1 j_2 \ldots j_n} \sqrt{g \circ \mathbf{p}(t, t_0, \mathbf{x}_0)}\ \mathrm{d}x_0^{j_1} \wedge \mathrm{d}x_0^{j_2} \wedge \cdots \wedge \mathrm{d}x_0^{j_n}, \end{aligned} \tag{4}$$

where

$$\det(A) = \varepsilon_{i_1 i_2 \cdots n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

has been used.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \sqrt{g(\mathbf{x})}\,\rho(t,\mathbf{x}) \\
= \;& \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \mathrm{d}x_0^1 \wedge \mathrm{d}x_0^2 \wedge \cdots \wedge \mathrm{d}x_0^n \, J\,(\sqrt{g}\rho)\circ \mathbf{p} \\
= \;& \int_{\Omega_0} \mathrm{d}x_0^1 \wedge \mathrm{d}x_0^2 \wedge \cdots \wedge \mathrm{d}x_0^n \, \frac{\partial}{\partial t}\left[J\,(\rho\sqrt{g})\circ \mathbf{p}\right] \\
= \;& \int_{\Omega_0} \mathrm{d}x_0^1 \wedge \mathrm{d}x_0^2 \wedge \cdots \wedge \mathrm{d}x_0^n \left[\frac{\partial}{\partial t}\left(\sqrt{g\circ\mathbf{p}}\,J\right)\,\rho\circ\mathbf{p} + J\sqrt{g\circ\mathbf{p}}\,\left(\frac{\partial\rho}{\partial t} + V^k\frac{\partial\rho}{\partial x^k}\right)\circ\mathbf{p}\right] \\
= \;& \int_{\Omega_0} \mathrm{d}x_0^1 \wedge \mathrm{d}x_0^2 \wedge \cdots \wedge \mathrm{d}x_0^n \, J\sqrt{g\circ\mathbf{p}}\left(\rho\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}V^k\right) + \frac{\partial\rho}{\partial t} + V^k\frac{\partial\rho}{\partial x^k}\right)\circ\mathbf{p} \\
= \;& \int_{\Omega_t} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \sqrt{g}\,\left[\rho\,\mathrm{div}\,\mathbf{V} + \frac{\partial\rho}{\partial t} + \mathbf{V}\cdot\nabla\rho\right] \\
= \;& \int_{\Omega_t} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \sqrt{g}\,\left[\frac{\partial\rho}{\partial t} + \mathrm{div}\,\left(\rho\mathbf{V}\right)\right]. \qquad (5)
\end{aligned}$$

Time derivative of Jacobian,

$$\begin{aligned}
\mathbf{A}^{-1}\mathbf{A} &= \mathbf{I}, \quad \mathbf{C}^T\mathbf{A} = \det(\mathbf{A})\mathbf{I}, \quad J\,\delta^i{}_k = c_j{}^i a^j{}_k, \\
\frac{\partial}{\partial t}J &= \frac{\partial}{\partial t}\left(c_j{}^{(i)} a^j{}_{(i)}\right) = c_j{}^i\frac{\partial}{\partial t}\left(a^j{}_i\right) = c_j{}^i\frac{\partial}{\partial t}\left(\frac{\partial p^j}{\partial x_0^i}\right) \\
&= c_j{}^i\frac{\partial}{\partial x_0^i}\frac{\partial p^j}{\partial t} = c_j{}^i\frac{\partial}{\partial x_0^i}\left(V^j\circ\mathbf{p}\right) = c_j{}^i\left(\frac{\partial V^j}{\partial x^k}\circ\mathbf{p}\right)\frac{\partial p^k}{\partial x_0^i} \\
&= c_j{}^i a^k{}_i\frac{\partial V^j}{\partial x^k}\circ\mathbf{p} = J\,\delta_j^k\frac{\partial V^j}{\partial x^k}\circ\mathbf{p} = J\frac{\partial V^i}{\partial x^i}\circ\mathbf{p}. \qquad (6) \\
\frac{\partial}{\partial t}\left(\sqrt{g\circ\mathbf{p}}\right) &= \left(V^k\frac{\partial}{\partial x^k}\sqrt{g}\right)\circ\mathbf{p} \qquad (7) \\
\frac{\partial}{\partial t}\left(\sqrt{g\circ\mathbf{p}}\,J\right) &= \sqrt{g\circ\mathbf{p}}\,J\left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}V^k\right)\right] = J\left(\sqrt{g}\,\mathrm{div}\,\mathbf{V}\right)\circ\mathbf{p}. \qquad (8)
\end{aligned}$$

Change of differential volume valid for the flow of any time dependent vector field

$$\frac{\partial}{\partial t}\left(\sqrt{g\circ \mathbf{p}}\, J\right) \;=\; J\left(\sqrt{g}\,\mathrm{div}\,\mathbf{V}\right)\circ \mathbf{p}. \qquad (9)$$

Conservation of total mass implies

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{V}) \;=\; 0. \qquad (10)$$

Liouville's theorem says that in case of zero divergence of the vector field $\mathbf{V}$, the mass density along field lines is constant,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\circ \mathbf{p}) \;=\; \frac{\partial \rho}{\partial t} + \mathbf{V}\cdot\nabla\rho \;=\; 0. \qquad (11)$$

The theorem also says that phase space volume is conserved even in the case of a time dependent vector field.

In kinetic plasma theory, particle conservation governed by the single particle distribution function

$$\int \mathrm{d}x^3 \mathrm{d}v^3 f(t,\mathbf{x},\mathbf{v}) \;=\; N, \qquad (12)$$

and the divergence free Hamiltonian vector field Vlasov*s equation is obtained

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} \;=\; 0. \qquad (13)$$

Expressed by Liouville's theorem, the distribution function is constant along the integral lines of Hamiltonian vector fields.

The characteristic equations are

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v}, \\ \dot{\mathbf{v}} &= \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \end{aligned}$$

and, therefore,

$$\mathrm{div}\,\mathbf{V} \;=\; \frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{v} + \frac{\partial}{\partial \mathbf{v}}\cdot\frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) \;=\; 0,$$

because

$$\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}\times\mathbf{B}) \;=\; 0.$$