Summary Sec. 1.2.1:
The non-interacting Green's function

Starting point: the general electron Green's function (1.1):

$$iG_{\alpha,\beta}(\mathbf{r}t,\mathbf{r}'t') = < \Psi_0^{(N)} | \hat{T} \left[\hat{\psi}_{H\alpha}(\mathbf{r}t)\, \hat{\psi}^\dagger_{H\beta}(\mathbf{r}'t') \right] | \Psi_0^{(N)} >$$

A perturbation calculation:

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

Operator transformation: Schrödinger $\longrightarrow$ Heisenberg
Schrödinger $\longrightarrow$ Interaction

$$\hat{O}_H(t) = \mathrm{e}^{i\hat{H}t/\hbar}\, \hat{O}\, \mathrm{e}^{-i\hat{H}t/\hbar}$$

$$\hat{O}_I(t) = \mathrm{e}^{i\hat{H}_0t/\hbar}\, \hat{O}\, \mathrm{e}^{-i\hat{H}_0t/\hbar}$$

Equation of motion of an Interaction Operator:

$$i\hbar \frac{\partial}{\partial t} \hat{O}_I(t) = \mathrm{e}^{i\hat{H}_0t/\hbar} \left[\hat{O}, \hat{H}_0 \right] \mathrm{e}^{-i\hat{H}_0t/\hbar}$$

The non-interacting Hamiltonian in the
particle number representation

$$\hat{H}_0 = \sum_{i,j} < i|\hat{h}|j > \hat{c}_i^\dagger \hat{c}_j$$

is diagonal in its eigenbasis $\quad \hat{h}|i> = \hbar\omega_i^0 |i>$

$$\hat{H}_0 = \sum_i \hbar\omega_i^0\, \hat{c}_i^\dagger \hat{c}_i$$

Equation of motion of the annihilation operator:

$$i\hbar \frac{\partial}{\partial t} \hat{c}_{jI}(t) = \mathrm{e}^{i\hat{H}_0t/\hbar} \left[\hat{c}_j, \hat{H}_0 \right] \mathrm{e}^{-i\hat{H}_0t/\hbar},$$

and with respect of $\left[\hat{c}_j, \hat{H}_0 \right] = \hbar\omega_j^0\, \hat{c}_j \quad :$

$$i\hbar \frac{\partial}{\partial t} \hat{c}_{jI}(t) = \hbar\omega_j^0\, \hat{c}_{jI}(t)$$

Solution of this differential equation:

$$\hat{c}_{jI}(t) = \hat{c}_j \, \mathrm{e}^{-i\omega_j^0 t}$$

and -correspondingly- for the creation operator:

$$\hat{c}_{jI}^\dagger(t) = \hat{c}_j^\dagger \, \mathrm{e}^{+i\omega_j^0 t}$$

Now back to the Green's function of the non-interacting electron:

$$iG^0_{\alpha,\beta}(\mathbf{r}t, \mathbf{r}'t') =< \Phi_0|\hat{T}\left[\hat{\psi}_{H\alpha}(\mathbf{r}t)\, \hat{\psi}^\dagger_{H\beta}(\mathbf{r}'t')\right]|\Phi_0 >$$

with

$$\hat{H} = \hat{H}_0 \qquad \text{and} \qquad \hat{H}_0|\Phi_0 >= E_0|\Phi_0 >$$

Important consequence: "Heisenberg = Interaction":

$$\hat{O}_H(t) = \hat{O}_I(t)$$

Therefore one gets

$$iG^0_{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') =< \Phi_0|\hat{T}\left[\hat{\psi}_{\alpha I}(\mathbf{r}t)\hat{\psi}^\dagger_{\beta I}(\mathbf{r}'t')\right]|\Phi_0 > \ .$$

Expansion of the field operators with respect of the annihilation and creation operators:

$$\hat{\psi}_{\alpha I}(\mathbf{r}t) = \sum_{\mathbf{k}\lambda} \psi_{\mathbf{k}}(\mathbf{r})\chi_\lambda(\alpha)\, \hat{c}_{\mathbf{k}\lambda,I}(t) = \sum_{\mathbf{k}\lambda} \psi_{\mathbf{k}}(\mathbf{r})\chi_\lambda(\alpha)\, \mathrm{e}^{-i\omega^0_{\mathbf{k}}t}\, \hat{c}_{\mathbf{k}\lambda}$$

and

$$\hat{\psi}^\dagger_{\beta I}(\mathbf{r}'t') = \sum_{\mathbf{k}\lambda} \psi^*_{\mathbf{k}}(\mathbf{r}')\chi^\dagger_\lambda(\beta)\, \hat{c}^\dagger_{\mathbf{k}\lambda,I}(t) = \sum_{\mathbf{k}\lambda} \psi^*_{\mathbf{k}}(\mathbf{r}')\chi^\dagger_\lambda(\beta)\, \mathrm{e}^{+i\omega^0_{\mathbf{k}}t'}\, \hat{c}^\dagger_{\mathbf{k}\lambda}\, .$$

For $T \to 0$ K, the ground state of a system of non-interacting electrons is given by a completely filled Fermi sphere with radius k_F.
For practical reasons, the operator $\hat{c}$ is re-defined by the following canonical transformation:

$$\hat{c}_{\mathbf{k}\lambda} = \begin{cases} \hat{a}_{\mathbf{k}\lambda} & \text{"annihilation of a particle state" for } \quad |\mathbf{k}| > k_F \\ \hat{b}^\dagger_{-\mathbf{k}\lambda} & \text{"creation of a hole state" for } \quad |\mathbf{k}| < k_F \end{cases}$$

Including this definition into the formula for the field operator $\hat{\psi}_{\alpha I}$ leads to

$$\hat{\psi}_{\alpha I}(\mathbf{r}t) = \sum_{|\mathbf{k}|>k_F} \sum_{\lambda} \psi_{\mathbf{k}}(\mathbf{r})\chi_\lambda(\alpha)\, \mathrm{e}^{-i\omega_{\mathbf{k}}^0 t}\, \hat{a}_{\mathbf{k}\lambda} + \sum_{|\mathbf{k}|<k_F} \sum_{\lambda} \psi_{\mathbf{k}}(\mathbf{r})\chi_\lambda(\alpha)\, \mathrm{e}^{-i\omega_{\mathbf{k}}^0 t}\, \hat{b}^\dagger_{-\mathbf{k}\lambda} \,.$$

Now such equations for $\hat{\psi}_{\alpha I}$ and $\hat{\psi}^\dagger_{\beta I}$ are used in the expression of $G^0_{\alpha\beta}$. In case of $t > t'$, one gets the following four matrix elements:

1. $< \Phi_0|\hat{a}_{\mathbf{k}\lambda}\hat{a}^\dagger_{\mathbf{k}'\lambda'}|\Phi_0 >= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$

2. $< \Phi_0|\hat{a}_{\mathbf{k}\lambda}\hat{b}_{-\mathbf{k}'\lambda'}|\Phi_0 > = 0$

3. $< \Phi_0|\hat{b}^\dagger_{-\mathbf{k}\lambda}\hat{a}^\dagger_{\mathbf{k}'\lambda'}|\Phi_0 > = 0$

4. $< \Phi_0|\hat{b}^\dagger_{-\mathbf{k}\lambda}\hat{b}_{-\mathbf{k}'\lambda'}|\Phi_0 > = 0$

and

$$\begin{aligned} iG^0_{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t') &= \sum_{|\mathbf{k}|>k_F} \psi_{\mathbf{k}}(\mathbf{r})\psi^*_{\mathbf{k}}(\mathbf{r}')\, \mathrm{e}^{-i\omega^0_{\mathbf{k}}(t-t')} \underbrace{\sum_\lambda \chi_\lambda(\alpha)\chi^\dagger_\lambda(\beta)}_{\delta_{\alpha,\beta}} \\ &= \delta_{\alpha,\beta} \sum_{|\mathbf{k}|>k_F} \psi_{\mathbf{k}}(\mathbf{r})\psi^*_{\mathbf{k}}(\mathbf{r}')\, \mathrm{e}^{-i\omega^0_{\mathbf{k}}(t-t')} \end{aligned}$$

The correspondent result for $t < t'$ reads

$$iG^0_{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t') = -\delta_{\alpha,\beta} \sum_{|\mathbf{k}|<k_F} \psi_{\mathbf{k}}(\mathbf{r})\psi^*_{\mathbf{k}}(\mathbf{r}')\, \mathrm{e}^{-i\omega^0_{\mathbf{k}}(t-t')}$$

leading to

$$iG^0_{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t') = \delta_{\alpha\beta} \sum_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r})\psi^*_{\mathbf{k}}(\mathbf{r}')\, \mathrm{e}^{-i\omega^0_{\mathbf{k}}(t-t')}$$
$$\times \left\{ \Theta(t-t')\Theta(|\mathbf{k}|-k_F) - \Theta(t'-t)\Theta(k_F-|\mathbf{k}|) \right\} \,.$$

By using the integral representation

$$\Theta(t-t') = -\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega(t-t')}}{\omega + i\eta}$$

for the time-dependent Heaviside function, one finally gets (see pp. 8 and 9 of my lecture notes)

$$G^0_{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t') = \delta_{\alpha\beta} \sum_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r})\psi^*_{\mathbf{k}}(\mathbf{r}') \int \frac{d\sigma}{2\pi} e^{-i\sigma(t-t')} \left\{ \frac{\Theta(k-k_F)}{\sigma - \omega^0_{\mathbf{k}} + i\eta} + \frac{\Theta(k_F-k)}{\sigma - \omega^0_{\mathbf{k}} - i\eta} \right\} .$$

Choice of the one-particle eigenbasis:

$$\hat{h}|i>= \hbar\omega^0_i|i> \quad \rightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + v(\mathbf{r}) \right] \psi_{\mathbf{k}}(\mathbf{r}) = \hbar\omega^0_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{r})$$

with $v(\mathbf{r})$ as a local, external potential energy that acts on the electron.

The special choice:
homogeneous electron gas (jellium) $\rightarrow \quad v(\mathbf{r}) \equiv 0$:

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}} \quad \textbf{with} \quad \omega^0_{\mathbf{k}} = \frac{\hbar k^2}{2m}$$

leads to the non-interacting Green's function

$$G^0_{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t') = \frac{\delta_{\alpha\beta}}{\Omega} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \left\{ \frac{\Theta(k-k_F)}{\omega - \omega^0_{\mathbf{k}} + i\eta} + \frac{\Theta(k_F-k)}{\omega - \omega^0_{\mathbf{k}} - i\eta} \right\} .$$

Obviously, this result is

- **diagonal in spin space (because of $\delta_{\alpha,\beta}$)**
- **homogeneous in space and time:**

$$G^0_{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t') = \delta_{\alpha\beta}\, G^0(\mathbf{r}-\mathbf{r}';t-t') .$$

An enormous advantage of this homogeneity is that one can easily change from the $\{\mathbf{r};t\}$ space to the $\{\mathbf{k};\omega\}$ space by applying the Fourier transform

$$G^0_{\alpha\beta}(\mathbf{r}-\mathbf{r}';t-t') = \frac{1}{\Omega}\sum_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int \frac{d\omega}{2\pi}\,\mathrm{e}^{-i\omega(t-t')}\,G^0_{\alpha\beta}(\mathbf{k},\omega)$$

with

$$G^0_{\alpha\beta}(\mathbf{k},\omega) = \delta_{\alpha\beta}\,G^0(\mathbf{k},\omega)$$

and

$$G^0(\mathbf{k},\omega) = \frac{\Theta(k-k_F)}{\omega-\omega^0_{\mathbf{k}}+i\eta} + \frac{\Theta(k_F-k)}{\omega-\omega^0_{\mathbf{k}}-i\eta}\,.$$