Computer Tomography: A Short Summary

1 Introduction

Fourier analysis plays an enormous role in many fields of physics. In the following, we would like to discuss the problem of *computer tomography* (CT), a method to reconstruct two- (2D) or three-dimensional (3D) distributions of various physical quantities from a finite number of one-dimensional (1D) *profiles* by using the method of *discrete Fourier transformation.* Nowadays, such methods of reconstruction have become more and more popular in Solid State Physics, e. g., to obtain 3D momentum densities of electrons in crystals on the basis of experimentally yielded 1D *Compton profiles.* Especially important is the method of CT in the medical diagnosis, in order to get 3D informations about the interior of the human body based on a series of 2D X-ray pictures.

The mathematical background for computer tomography has been created by *Cormack* and *Hounsfield* who were honored by the Nobel Prize for medicine in 1979.

There is a number of mathematically different methods for CT reconstructions, but the most important one is based on Fourier transformations. The principles of this method are described in numerous textbooks and shall not be repeated here. The following explanations concerning CT are mainly taken out of the textbook of P. L. DeVries, *A First Course in Computational Physics*, John Wiley, Inc., New York, 1994.

2 Theoretical background of the program

The principle of the geometry of CT is presented in Fig. 1. An object with the mass distribution $f(x, y)$ is situated in the $(x; y)$ space and is penetrated by an X-ray (dotted line) whose position and direction are determined by the angle ϕ and the minimal distance ξ from the origin of the coordinate system. While penetrating the object, the intensity I of the X-ray is reduced according to

$$I = I_0 \, \mathrm{e}^{-\int d\eta \, f(x,y)} \, ,$$

where $d\eta$ means a differential element of the way of the X-ray through the object. The (negative) logarithm of the ratio I/I_0,

$$p(\xi; \phi) = -\ln\left(\frac{I}{I_0}\right) = \int d\eta \, f(x, y) \, , \tag{1}$$

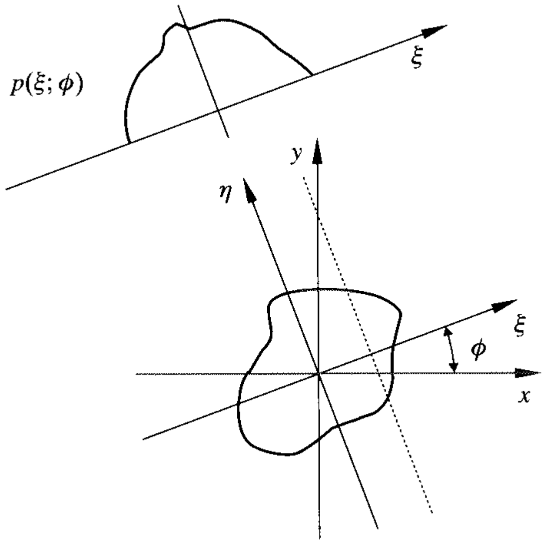

Figure 1: Geometry of CT: a *projection function (profile)* $p(\xi, \phi)$ of the object is obtained if a series of X-rays penetrates it parallel to the axis η and perpendicular to the axis ξ, where ϕ means the angle between the direction ξ and the axis x of the fixed Cartesian system $(x; y)$.

is called the *projection function* $p(\xi, \phi)$. If one uses a number of parallel rays with constant ϕ and different values of ξ – equally distributed over the interval $[-\xi_{max}, \xi_{max}]$ –, one obtains a one-dimensional *profile* of the object for the angle ϕ.

The main goal of CT is to reconstruct the 2D mass distribution $f(x, y)$ using a finite number of profiles $p(\xi, \phi)$.

According to Fig. 1, there exist two coordinate systems: the first one $(x; y)$ is firmly connected with the *scanned* object, and the second one $(\xi; \eta)$ rotates by the angle ϕ. The relation between these systems is simply given by

$$\xi = x \cos\phi + y \sin\phi \qquad \text{and} \qquad \eta = -x \sin\phi + y \cos\phi \,. \tag{2}$$

The first step of the calculation is a Fourier transformation (FT) of the function f from real space $(x; y)$ in the reciprocal space $(k_x; k_y)$:

$$F(k_x, k_y) = \frac{1}{2\pi} \int \int_{-\infty}^{+\infty} dx dy \, f(x, y) \, \mathrm{e}^{i(k_x x + k_y y)} \,. \tag{3}$$

According to Fig. 2, we can also define the wave vector $\mathbf{k}$ in the $(\xi; \eta)$ system by

$$k_\xi = k_x \cos\phi + k_y \sin\phi \qquad \text{and} \qquad k_\eta = -k_x \sin\phi + k_y \cos\phi \,, \tag{4}$$

and it can easily be shown that the exponential factor in Eq. (3) has the property

$$\mathrm{e}^{i(k_x x + k_y y)} = \mathrm{e}^{i(k_\xi \xi + k_\eta \eta)} \,. \tag{5}$$

Now to an important point: we do not calculate the FT of $f(x, y)$ for the whole $(k_x; k_y)$ space, but only along the k_ξ axis (i. e., for $k_\eta = 0$); consequently, with

$$k_x = k_\xi \cos\phi \qquad \text{and} \qquad k_y = k_\xi \sin\phi \,,$$

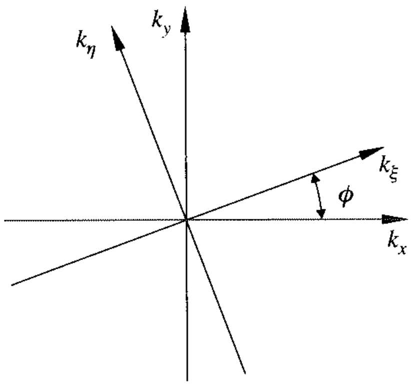

Figure 2: The coordinate systems in $\mathbf{k}$ space corresponding to Fig. 1.

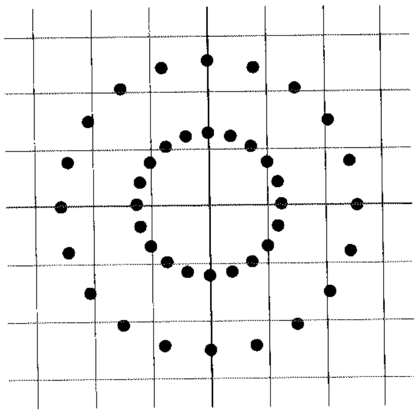

Figure 3: Distribution of the Fourier coefficients of the mass distribution function $f(x, y)$ in the $(k_x; k_y)$ space, as obtained by Eq. (7).

we get

$$F(k_\xi \cos\phi, k_\xi \sin\phi) = \frac{1}{2\pi} \int\int_{-\infty}^{+\infty} dxdy\, f(x, y)\, \mathrm{e}^{ik_\xi \xi}\,.$$

Taking into account that $dxdy = d\xi d\eta$, the last expression can be transformed into

$$F(k_\xi \cos\phi, k_\xi \sin\phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \left(\int_{-\infty}^{+\infty} d\eta\, f(x, y) \right) \mathrm{e}^{ik_\xi \xi}\,, \qquad (6)$$

and a comparison between Eqs. (6) and (1) leads to the important equation

$$F(k_\xi \cos\phi, k_\xi \sin\phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi\, p(\xi; \phi)\, \mathrm{e}^{ik_\xi \xi}\,. \qquad (7)$$

This equation represents a relation between the (measured) profiles and the Fourier coefficients of the wanted mass distribution for arguments $(k_x; k_y)$ along *radial* lines in the $\mathbf{k}$ space (Fig. 3).

We are now confronted with a problem: in order to determine the mass distribution function $f(x, y)$ from its Fourier coefficients $F(k_x, k_y)$, an inverse FT has to be performed. Such a calculation would be relatively easy if F would be known for arguments $(k_x; k_y)$ arranged *on a Cartesian point lattice.* However, as we see from Fig. 3, this is obviously not the case! A transformation of the Fourier coefficients from *polar* coordinates to *Cartesian* coordinates by interpolation would be problematic because such interpolations would cause *local errors* in $\mathbf{k}$ space, and during the following inverse FT, these errors would *spread out without control* over the $(x; y)$ space.

Therefore, a *direct inverse FT* is rarely applied to CT. Instead of this, a special method of back transformation is used where all (unavoidably) interpolations can be done in *real space.*

How can we do that? Let's start from the inverse FT which belongs to Eq. (3):

$$f(x,y) = \frac{1}{2\pi} \int\int_{-\infty}^{+\infty} dk_x dk_y \, F(k_x, k_y) \, \mathrm{e}^{-i(k_x x + k_y y)} \, . \tag{8}$$

Remember: the arguments $(k_x; k_y)$ lie on the axis k_ξ; we have therefore

$$k_x = \rho \cos\phi \qquad \text{and} \qquad k_y = \rho \sin\phi$$

with $\rho = \sqrt{k_x^2 + k_y^2}$ and

$$x = r\cos\Theta \qquad \text{and} \qquad y = r\sin\Theta$$

leading to (see Fig. 4)

$$k_x x + k_y y = \rho r \cos\phi\cos\Theta + \rho r \sin\phi \sin\Theta = \rho r \cos(\Theta - \phi) \, . \tag{9}$$

Taking this into account, Eq. (8) takes the form

$$\begin{aligned}
f(r\cos\Theta, r\sin\Theta) &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} d\phi d\rho \, \rho \, F(\rho\cos\phi, \rho\sin\phi) \, \mathrm{e}^{-i\rho r\cos(\Theta-\phi)} \\
&= \frac{1}{2\pi}\int_0^\infty \int_0^{\pi} d\phi d\rho \, \rho \, F(\rho\cos\phi, \rho\sin\phi) \, \mathrm{e}^{-i\rho r\cos(\Theta-\phi)} + \\
&\quad + \frac{1}{2\pi}\int_0^\infty \int_\pi^{2\pi} d\phi d\rho \, \rho \, F(\rho\cos\phi, \rho\sin\phi) \, \mathrm{e}^{-i\rho r\cos(\Theta-\phi)} \\
&= \frac{1}{2\pi}\int_0^\infty \int_0^{\pi} d\phi d\rho \, \rho \, F(\rho\cos\phi, \rho\sin\phi) \, \mathrm{e}^{-i\rho r\cos(\Theta-\phi)} + \\
&\quad + \frac{1}{2\pi}\int_0^\infty \int_0^{\pi} d\phi d\rho \, \rho \, F(-\rho\cos\phi, -\rho\sin\phi) \, \mathrm{e}^{+i\rho r\cos(\Theta-\phi)} \\
&= \frac{1}{2\pi}\int_0^{\pi} d\phi \left\{ \int_0^\infty d\rho \, \rho \, F(\rho\cos\phi, \rho\sin\phi) \, \mathrm{e}^{-i\rho r\cos(\Theta-\phi)} + \right. \\
&\quad \left. + \int_0^\infty d\rho \, \rho \, F(-\rho\cos\phi, -\rho\sin\phi) \, \mathrm{e}^{+i\rho r\cos(\Theta-\phi)} \right\} \, .
\end{aligned} \tag{10}$$

Because of the fact that the vector $\mathbf{k}$ has only a component k_ξ, we can set $\rho = k_\xi$ in the last but one integral and $\rho = -k_\xi$ in the last integral. This leads to the result

$$f(r\cos\Theta, r\sin\Theta) = \frac{1}{2\pi}\int_0^\pi d\phi \int_{-\infty}^{+\infty} dk_\xi \, |k_\xi| \, F(k_\xi\cos\phi, k_\xi\sin\phi) \, \mathrm{e}^{-ik_\xi r\cos(\Theta-\phi)} \, ,$$

and, including the relation $\xi = r\cos(\Theta - \phi)$, we get

$$f(r\cos\Theta, r\sin\Theta) = \int_0^\pi d\phi \left\{ \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_\xi \, |k_\xi| \, F(k_\xi\cos\phi, k_\xi\sin\phi) \, \mathrm{e}^{-ik_\xi \xi} \right\} \, ,$$

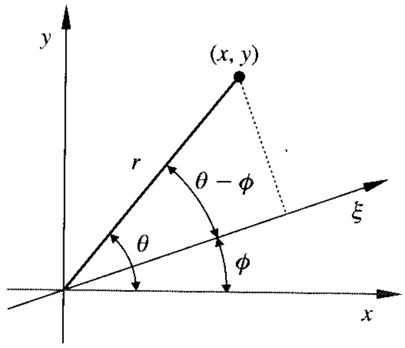

Figure 4: Explanation of Eq. (9).

The quantity within the brackets $\{\cdots\}$ is called the *modified projection* $\tilde{p}(\xi;\phi)$, and our **final result** reads

$$f(x,y) = f(r\cos\Theta; r\sin\Theta) = \int_0^{\pi} d\phi\, \tilde{p}(\xi;\Phi) \tag{11}$$

with

$$\tilde{p}(\xi;\phi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_\xi\, |k_\xi|\, F(k_\xi\cos\phi, k_\xi\sin\phi)\, \mathrm{e}^{-ik_\xi\xi} \tag{12}$$

and

$$\xi = r\cos(\Theta-\phi) = x\cos\phi + y\sin\phi\,. \tag{13}$$