Kap.5, Aufg.5: Charge Moving in the Field of a Magnetic Monopole.

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1. The Field

A real magnetic monopole has not been found. So this concept is an idealization of the field in the neighbourhood of one of the two poles of a very long magnetic dipole. The strength of a pole is g, the length of the dipole is a; the latter is assumed to be very large; this justifies the approximation introduced at the end of eq.(1) The magnetic field and induction are in this approximation:

$$\vec{H} = grad\Big(\frac{g}{r} - \frac{g}{|\vec{r}-\vec{a}|}\Big) = g\,\frac{\vec{r}}{r^3} - g\,\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3} \approx g\,\frac{\vec{r}}{r^3} \tag{1}$$

$$\vec{B} = \mu_0\,\vec{H} \approx \mu_0 g\,\frac{\vec{r}}{r^3} \tag{2}$$

2. The Solution of the Equation of Motion

The field is spherically symmetric. However, the Lorentz force containing it does not share this property. Nevertheless there are 5 integrals of the motion of a point particle with mass m and charge e in this field: Kinetic Energy $T = E$, $\vec{L}^2$ and the Poincaré Vektor $\vec{A}$. So the problem is completely integrable. The equation of motion is:

$$m\,\ddot{\vec{r}} = e\,(\vec{v}\times\vec{B}) = e\mu_0 g\,\Big[\dot{\vec{r}}, \frac{\vec{r}}{r^3}\Big]. \tag{3}$$

Multiplying eq.(3) with $\dot{\vec{r}}$ gives conservation of kinetic energy:

$$m\,(\ddot{\vec{r}}\cdot\dot{\vec{r}}) = 0; \quad \frac{d}{dt}\frac{m}{2}\dot{\vec{r}}^2 = 0, \quad E = T = \frac{m}{2}\dot{\vec{r}}^2 = const. \tag{4}$$

Multiplying eq.(3) with $\vec{r}\times$ gives the following time derivatives:

$$m\,(\vec{r}\times\ddot{\vec{r}}) = \underline{\frac{d}{dt}\vec{L}} = \frac{e\mu_0 g}{r^3}\,(\vec{r}\times(\dot{\vec{r}}\times\vec{r})) = \vec{M} \tag{5}$$

$$= e\mu_0 g\,\Big(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}(\vec{r}\cdot\dot{\vec{r}})}{r^3}\Big)$$

$$= \underline{\frac{d}{dt}\Big(e\mu g\,\frac{\vec{r}}{r}\Big)}\,. \tag{6}$$

The scalar product of eq.(5) with $\vec{L}$ gives the conservation of $\vec{L}^2$:

$$\vec{L}\cdot\frac{d\vec{L}}{dt} = \frac{d}{dt}\frac{\vec{L}^2}{2} = 0; \qquad \vec{L}^2 = const. \tag{7}$$

Taking the underlined parts of eqs.(5) and (6) gives:

$$\frac{d}{dt}\Big[\vec{L} - e\mu_0 g\,\frac{\vec{r}}{r}\Big] = 0; \quad \vec{A} := \vec{L} - e\mu_0 g\,\frac{\vec{r}}{r} = const. \tag{8}$$

This conserved vector was found by Poincaré. Multiplying it with $\vec{r}/r$ gives a condition for the orbit ($\vec{r}\cdot\vec{L}=0$ is used !) :

$$\left(\vec{A}\cdot\frac{\vec{r}}{r}\right) \;=\; -e\mu_0 g \;=\; A\ \cos(\vec{A}\cdot\vec{r}) \;=\; const. \tag{9}$$

$$\cos(\vec{A}\cdot\vec{r}) \;=\; -e\mu_0 g/A \;=\; \frac{-e\mu_0 g}{\sqrt{L^2+(e\mu_0 g)^2}} \;=\; const. \tag{10}$$

The orbit is confined to a cone with apical angle:

$$\vee(\vec{A},\vec{r}) \;=\; \arccos(-e\mu_0 g/A) \;:=\; \vartheta_A \;=\; const. \tag{11}$$

Now spherical coordinates r, ϑ, φ are introduced, whose centre is located in the monopole, whose polar axis $\vartheta = 0$ coincides with the vector $\vec{A}$. So the variable $\vartheta = \vartheta_A$ is constant and equals the apical angle. In these coordinates we get for total energy and for the square of angular momentum:

$$E \;=\; T \;=\; \frac{m}{2}\,\vec{v}^2 \;=\; \frac{m}{2}\Big(\dot{r}^2 \;+\; \frac{L^2}{m^2r^2}\Big) \;=\; \frac{m}{2}\,\vec{v}_0^2 \;=\; const. \tag{12}$$

$$L^2 \;=\; m^2\, r^4\ \sin^2\vartheta_A\ \dot{\varphi}^2 \;=\; (m\ \vec{r}_0\ \times\ \vec{v}_0)^2 \;=\; const. \tag{13}$$

From (12) follows that the radius is not smaller than r_m:

$$r \;\geq\; r_m \;=\; \frac{L}{\sqrt{2mE}} \;=\; \frac{L}{p_0}. \tag{14}$$

$\sqrt{2mE}$ equals the initial linear momentum p_0. Solving (12) for $\dot{r}$ we get a differential equation for radial motion (which can be solved by separation of variables and the substitution $u = r^2$):

$$\dot{r} \;=\pm\sqrt{2E/m}\ \sqrt{1-r_m^2/r^2}, \qquad r^2 \;=\; r_m^2 \;+\; \frac{2E}{m}\ t^2 \;=\; r_m^2 \;+\; v_0^2\ t^2. \tag{15}$$

Introducing spherical coordinates into the energy theorem (12) and inserting the conserved angular momentum (13) gives:

$$E \;=\; \frac{m}{2}\ v^2 \;=\; \frac{m}{2}\ \left(\frac{ds}{dt}\right)^2 \;=\; \frac{m}{2}\big[\dot{r}^2 \;+\; r^2\ \sin^2\vartheta_A\ \dot{\varphi}^2\big] \tag{16}$$

$$\;=\; \frac{m}{2}\ \dot{\varphi}^2\big[r'^2 \;+\; r^2\ \sin^2\vartheta_A\big] \;=\; \frac{L^2}{2m\sin^2\vartheta_A}\Big[\frac{r'^2}{r^4} \;+\; \frac{\sin^2\vartheta_A}{r^2}\Big] \tag{17}$$

In going to the second line above we changed the independent variable from time t to the azimuthal angle φ so that $r' = dr/d\varphi$. This differential equation is solved in the same way as that of the Kepler problem by use of the new dependent variable $s = 1/r$. The constant of integration occuring in this process is fixed such that $\varphi = 0$ for $r = r_m$. This gives the solution

$$r \;=\; \frac{\sin\vartheta_A}{A\ \cos(\sin\vartheta_A\ \varphi)} \;=\; \frac{r_m}{\cos(\sin\vartheta_A\ \varphi)} \tag{18}$$

The end points of the trajectory at infinity are obtained for:

$$r = \infty :\ \ \varphi\sin\vartheta_A = \pm\pi/2\ . \tag{19}$$

The trajectory is a screwlike curve lying entirely on the surface of the cone, whose apex is in the monopole, whose axis is given by the Poincaré vector $\vec{A}$ and whose apical angle is ϑ_A. A charged particle entering from infinity follows this narrowing spiralling path till $r = r_m$; there it is reflected back into infinity. So the radial magnetic field lines lying on this cone make up a cage for this charged particle.

This cage is completely closed for a pointlike pole. If the pole is extended then this conical cage does not end in a pointlike apex, but in a open tube, which permits some particles to escape. This is indeed the case in polar regions of the terrestrial magnetic field and in magnetic mirrors used for nuclear fusion.

This confinement by magnetic field lines is only approximately valid in the general case.