Nonequilibrium Green's functions

(Martin and Schwinger '59, Schwinger '61, Konstantinov and Perel '60, Dzyaloshinski (62), Kadanoff-Baym '62, Keldysh '64, ...)

Freely taken mainly from Rammer and Smith (1986) and Haug and Jauho (1998)

$$H = h + H'(t)$$

$$h = H_0 + V$$

↑ noninteracting

$$H'(t < t_0) = 0$$

FOR $t < t_0$ system is at equilibrium

with density matrix

$$\varrho(h) = \frac{1}{Z} e^{-\beta h}$$

Expectation values of operators for any t

$$\langle O(t) \rangle = \mathrm{Tr}\left(\varrho(h)\, O_H(t)\right)$$

↑ Heisenber repr. evolved according to H

Interaction representation

When introducing the interaction representation one should specify with respect to which "perturbation". Now we take

h as the unperturbed hamiltonian and $H'(t)$ as the perturbation

$$O_I(t) = e^{iht} O e^{-iht}$$

so it is the same as a Heisenberg representation with Hamiltonian h we therefore denote it as

$O_h(t)$ instead of $O_I(t)$

the reason is that later we will consider H_0 as the unperturbed Ham.

Similarly we denote the S - matrix as $S_h(t,t')$

Since for $t \leq t_0$ $\quad O_H(t) = O_h(t)$

we have $\quad O_H(t) = S_h^+(t,t_0)\, O_h(t)\, S_h(t,t_0)$

$$S_h(t,t_0) = T \exp\left(-i \int_{t_0}^{t} dt' H'_h(t')\right)$$

(we can as well consider $t_0 = -\infty$)

UNTIL HERE 23.3.11

We need expressions of the form (for simplicity we take zero temperature, and thus we have the ground state of h $|\Phi_0\rangle$,

(later we can take $t_0 = -\infty \Rightarrow |\Phi_0\rangle = |-\infty\rangle$)

$$\langle \Phi_0 | T A(t_1) B(t_2) \cdots | \Phi_0 \rangle$$

By proceeding as for the equilibrium case we get:

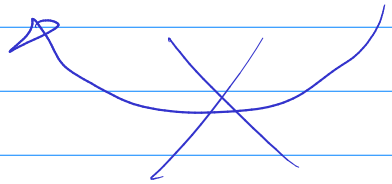
$$= \langle +\infty | T S_h(+\infty, t_0) A_h(t_1) B_h(t_2) \cdots | \Phi_0 \rangle$$

The crucial point is that now $|+\infty\rangle \neq$ PHASE $|\Phi_0\rangle$

The reason is that H' is not necessarily switched on adiabatically, neither it is switched off

We thus need to "go back" to t_0

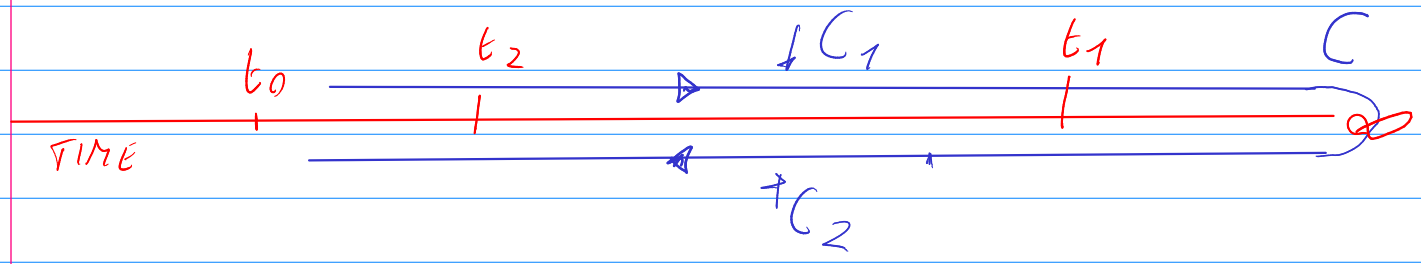
$$= \langle \Phi_0 | S_h(t_0, +\infty) T S_h(+\infty, t_0) A_h(t_1) B_h(t_2) \cdots | \Phi_0 \rangle$$



I cannot move the time ordering!

Notice, instead of ∞ , we could already "turn back" at max(t1,t2,..)

Trick: introduce contour C along the time axis "Keldysh" contour



now each operator must specify BOTH its time AND on which side of the contour it lays : C_1 OR C_2

$\gamma_A = (t_A, C_A)$ $\gamma_B = (t_B, C_B)$ ETC.

with an obvious definition of the ordering

$\gamma_A \geq_C \gamma_B$ IF EITHER $C_A = C_2$, $C_B = C_1$

$C_A = C_B = C_1$, $t_A > t_B$

$C_A = C_B = C_2$, $t_A < t_B$

Introduce contour ordering operator

$$T_C\, A(\gamma_A) B(\gamma_B) = \begin{cases} A(\gamma_A) B(\gamma_B) & (\gamma_A \geq_C \gamma_B) \\ -\varepsilon B(\gamma_B) A(\gamma_A) & (\gamma_B \geq_C \gamma_A) \end{cases}$$

we can now write

$$\langle \Phi_0 | S_h(t_0, +\infty)\, T\, S_h(+\infty, t_0) A_h(t_1) B_h(t_2) \cdots | \Phi_0 \rangle$$

$$= \langle \Phi_0 | T_C S_h(t_0 C_2, +\infty) S_h(+\infty, t_0 C_1) A_h(t_1 C_1) B_h(t_2 C_1) \cdots | \Phi_0 \rangle$$

notice that $+\infty$ is the turning point, so we do not need to specify on which part of the contour it is.

$$= \langle \Phi_0 | T_C\, S_h(t_0 C_2, t_0 C_1) A_h(t_1 C_1) B_h(t_2 C_1) \cdots | \Phi_0 \rangle$$

of course the operators A, B need not to be restricted to C1

so we can generalize the expression for a generic Green's function (which in the end we will need for perturbation theory)

$$\langle \Phi_0 | T_C\, A(\tau_1) B(\tau_2) \cdots | \Phi_0 \rangle =$$

$$= \langle \Phi_0 | T_C\, S_h(t_0 C_2, t_0 C_1) A_h(\tau_1) B_h(\tau_2) \cdots | \Phi_0 \rangle$$

where

$$S_h(t_0 C_2, t_0 C_1) = T_C\, e^{-i \int_C H_h'(\tau)\, d\tau}$$

We still cannot apply Wick's theorem, because h

is not quadratic. What to do ?

For zero temperature T=0 we only have the ground state

we can again think of switching on adiabatically the

~~interaction~~ at $t = -\infty$

Thus we can write

$$H = H_0 + \overbrace{V(t) + H'(t)}^{W(t)}$$

$$W(t) \propto \Theta(t - t_0)$$

$$= V e^{\alpha t} \qquad (\alpha \to 0 : \text{adiabatic})$$

we can now take $t_0 = -\infty$

$$h = H_0$$

$$H'(t) \longrightarrow H'(t) + V(t) \equiv W(t)$$

and apply formally the same results as above.

The difference is now that the Interaction representation

evolution and the initial state $|\Phi_0\rangle$

refer to a noninteracting (quadratic) hamiltonian
and thus Wick's theorem applies !

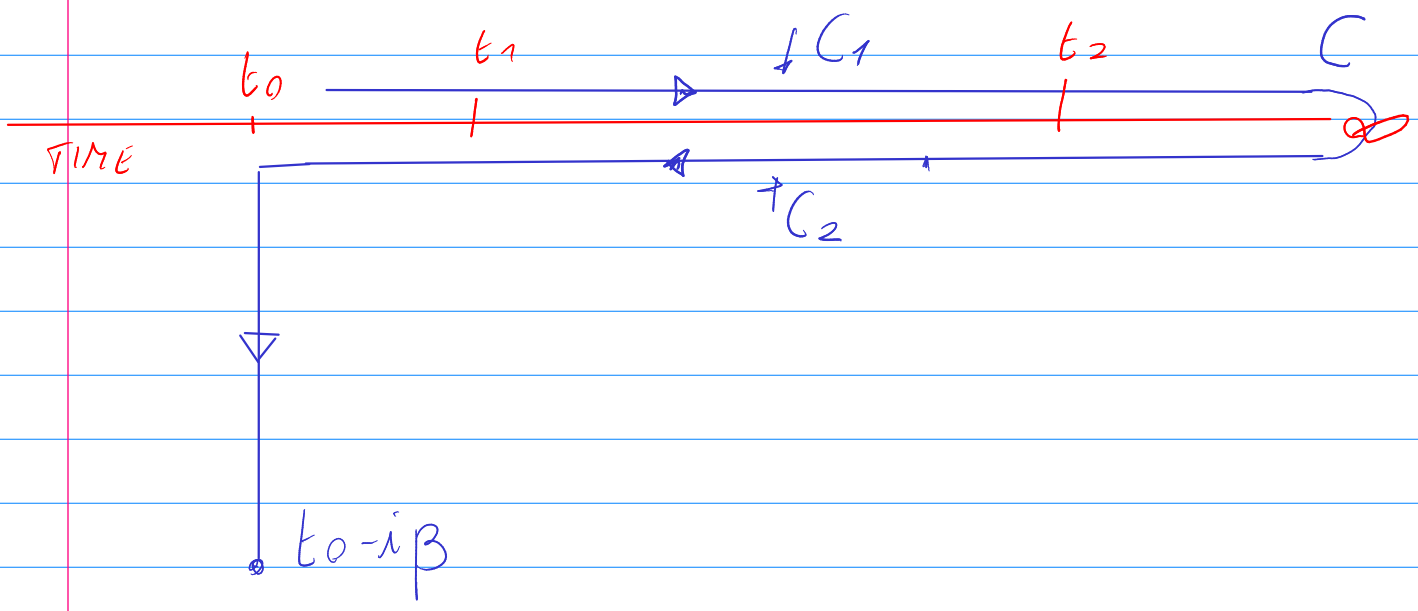
(Notice, by adiabatically switching on V , I guarantee that at the time in which I switch on H' (nonadiabatically) my state has evolved to the ground state of h)

FURTHER OPTIONS/ISSUES

WHICH WE ARE NOT GOING TO TREAT FOR THE MOMENT

START AT $t=t_0$ WITH AN ARBITRARY PREPARED STATE DEFINED BY A CORRELATED DENSITY MATRIX ρ

$\Rightarrow$ PATH ALONG IMAGINARY TIME TO TREAT INITIAL CORRELATIONS



It can be neglected when $t_0 \to -\infty$ and T=0

Besides, also an initial state with finite T at some t_0 can be produced with the help of a heat bath

Perturbation expansion

The nice point is that now perturbation expansion

in $W(t) \equiv V + H'(t)$

works very similarly to ordinary perturbation theory

Diagrammatic rules are the same

except that now all time integrals have to be carried out on the Keldysh contour C1+C2

Again the central object is the S Matrix

$$S \equiv S_{H_0}(-\infty_2, -\infty_1) = T_c \exp\left(-i \int_C W_{H_0}(t)\, dt\right)$$

interaction repr w.r. H_0.

Again one expands the exponential

$$S = T_c \left(1 - i \int_C W_{H_0}(\tau)\, d\tau - \frac{1}{2}\left(\int_C W_{H_0}(\tau)\, d\tau\right)^2 + \dots\right)$$

example: $-\frac{1}{2} T_c \int_C \int_C d\tau_1\, d\tau_2\, W_{H_0}(\tau_1)\, W_{H_0}(\tau_2)$

We need an expansion for objects of the form

$$\frac{\langle -\infty | T_c S A_{H_0}(t_1) B_{H_0}(t_2) \cdots | -\infty \rangle}{\langle -\infty | T_c S | -\infty \rangle}$$

The denominator is actually not necessary, because it is 1

However, one should use it for the linked-cluster theorem.
i.e. to cancel disconnected diagrams in the numerator

Each term of the expansion will contain expectation values in the noninteracting ground state $|-\infty\rangle$ of Tc ordered products of single-particle operators (i.e. creation and annihilation operators) time evolved according to H_0

Wick's theorem tells us that this can be decomposed into expectation values of time ordered pairs of products, i. e. Green's functions

$X_i = (\gamma_i$, other quantum numbers such as position, spin, orbital, etc. $)$

Example:

Notation:

$X_i \equiv (\gamma_i,$ other quantum numbers such as position, spin, orbital, etc. $)$

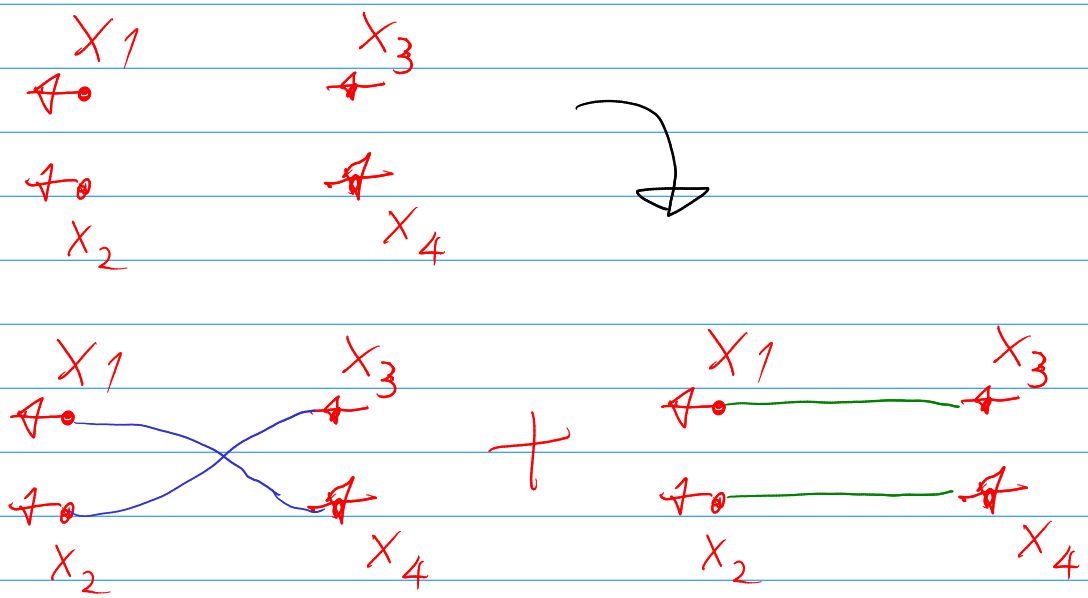
$\langle \cdots \rangle_0 \equiv \langle -\infty | \cdots | -\infty \rangle$ $\qquad C_0(X) \equiv C_{H_0}(X)$

$$\langle T_c \, C_0(X_1) C_0(X_2) C_0^\dagger(X_3) C_0^\dagger(X_4) \rangle_0$$

$$= \langle T_c \, C_0(X_1) C_0^\dagger(X_4) \rangle_0 \langle T_c \, C_0(X_2) C_0^\dagger(X_3) \rangle_0$$

$$- \xi \langle T_c \, C_0(X_1) C_0^\dagger(X_3) \rangle_0 \langle T_c \, C_0(X_2) C_0^\dagger(X_4) \rangle_0$$

Diagrammatically



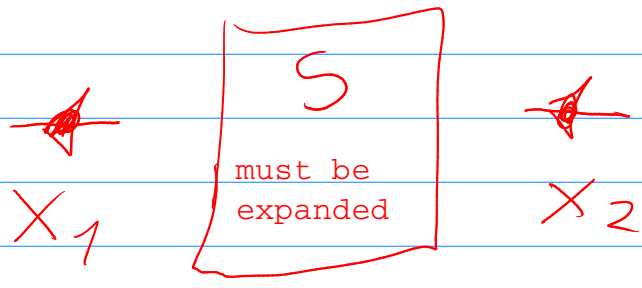
Example evaluation of a Green's function

$$G(x_1, x_2) = -i \langle T_c \, C(x_1) C^{+}(x_2) \rangle$$

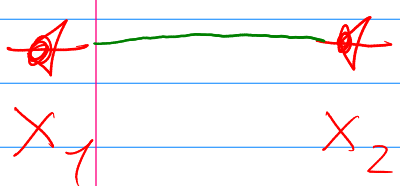
$$= \frac{-i \langle T_c \, S \, C_0(x_1) C_0^{+}(x_2) \rangle_0}{\langle T_c \, S \rangle_0}$$

Consider first numerator

x_1, x_2 External points



0th order $S^{(0)} = 1$



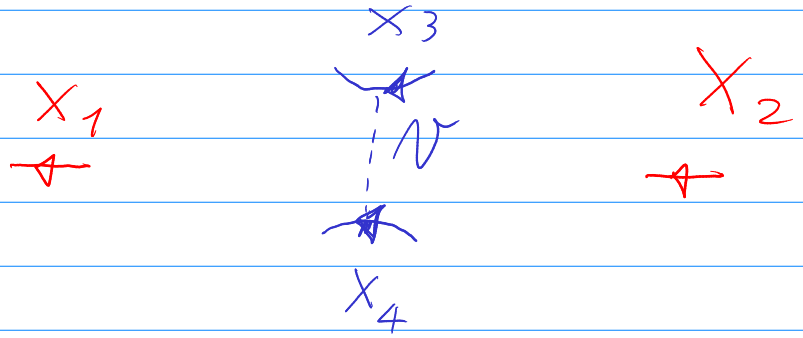
$$G_0(x_1, x_2) = -i \langle T_c C_0(x_1) C_0^{+}(x_2) \rangle_0$$

1st order (example, e. e. interaction)

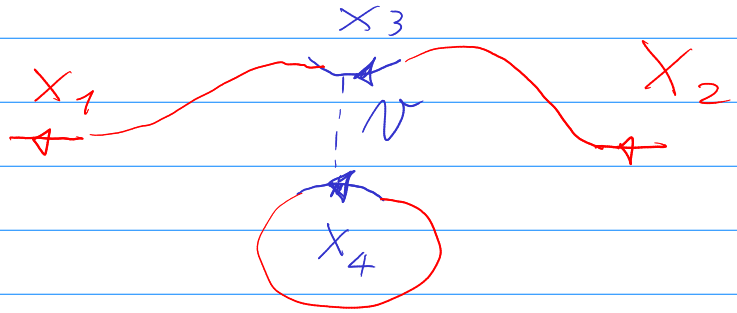
$$S^{(1)} \propto -\frac{i}{2}\int C_0^+(x_3)\,C_0^+(x_4)\,C_0(x_4)\,C_0(x_3)\,N\,V(x_3-x_4)\,dx_3\,dx_4$$

(x_3, x_4 ARE DIFFERENT BUT HAVE THE SAME TIME)

Represented by vertex



Wick's theorem: "all lines have to be paired in all possible ways"

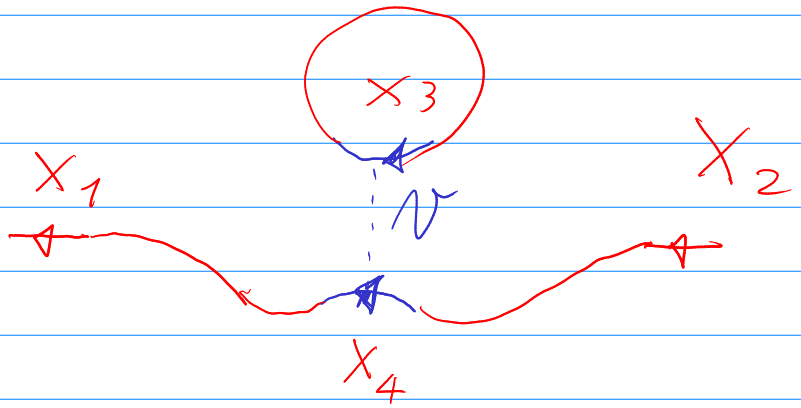


$$\propto \int dx_3\,dx_4\; G_0(x_1,x_3)\,G_0(x_3,x_2)\,G_0(x_4,x_4)\,V(x_3-x_4)$$

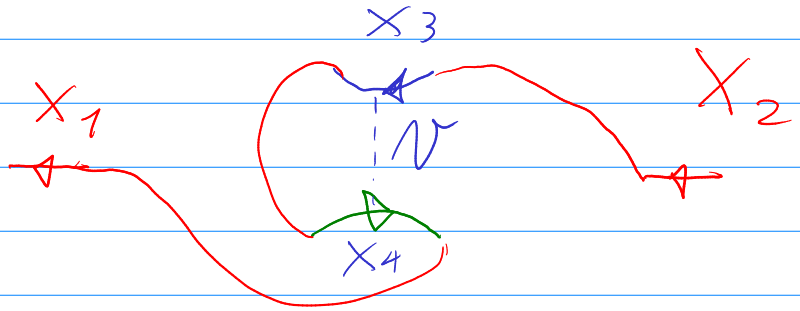
All internal variables X_3, X_4 are integrated or summed over

times are integrated over contour

X_1, X_2 being external variables are fixed



topologically equivalent to the former, just cancels factor 1/2

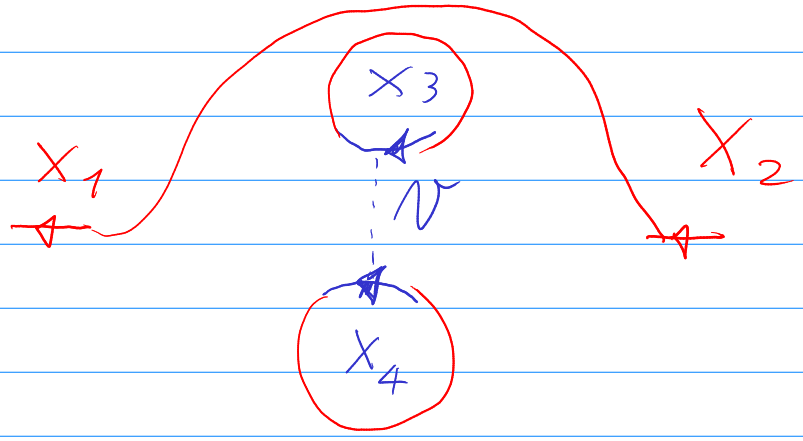


+ a topolgically equivalent one

$$\propto \int dX_3\, dX_4\, G_0(X_1, X_4)\, G_0(X_4, X_3)\, G_0(X_3, X_2)\, V(X_3 - X_4)$$

$$\equiv F(X_1, X_2)$$ (We will use it later)

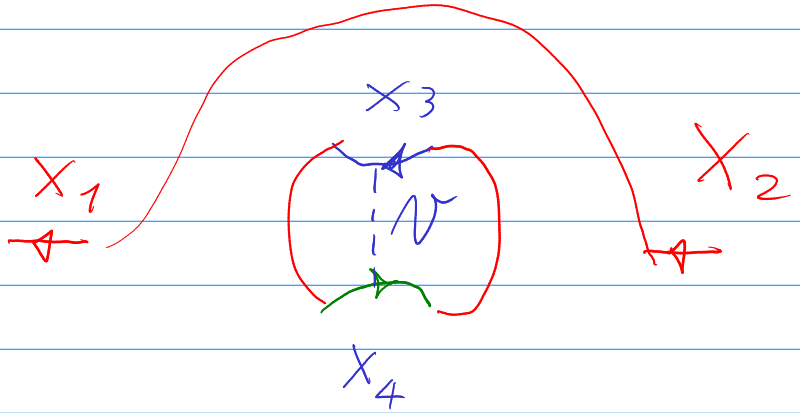
Disconnected diagrams:



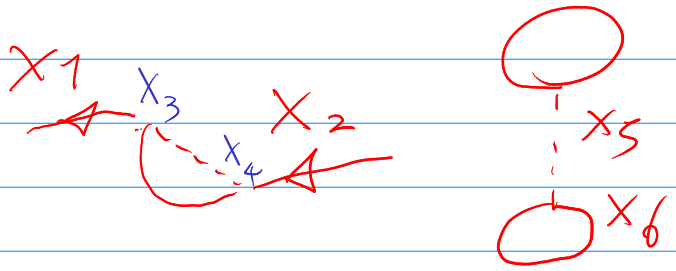
$\propto G_0(x_1 x_2) \cdot P$ factorize off

$$P = \int dx_3\, dx_4\, G_0(x_3, x_3)\, G_0(x_4 x_4)\, V(x_3 - x_4)$$

(P is calledd vacuum diagram)



Example of second order disconnected diagram



$\propto F(x_1, x_2) \cdot P$

Can be written as a product of lower-order diagrams !

Denominator $\langle T_c S \rangle_0$

only has vacuum diagrams such as P

Linked cluster theorem

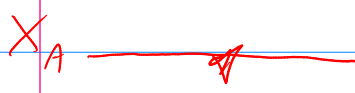
When evaluating numerator / denominator $\dfrac{-i \langle T_c S C_0(x_1) C_0^+(x_2) \rangle_0}{\langle T_c S \rangle_0}$

The effect of the denominator is simply to cancel out all disconnected diagram contributions !

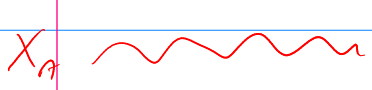
In the end one can restrict in evaluating CONNECTED diagrams

FEYNMAN DIAGRAMS RULES

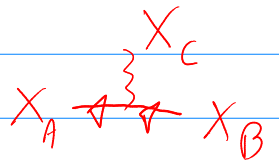
DIAGRAMMATIC ELEMENTS



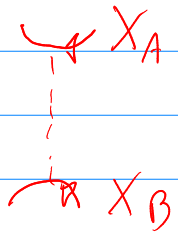
$= G_0(X_A, X_B)$ ELECTRON G.F.



$= D_0(X_A, X_B)$ PHONON G.F.



$= \mathcal{J}(X_A, X_B, X_C)$ ELECTR.-PHONON INTERACTION



$= V(X_A - X_B)$ ELECTRON-ELECTRON INTERACTION

$= U(X_A, X_B)$ SINGLE-PARTICLE POTENTIAL

DIAGRAMMATIC RULES (REAL SPACE AND TIME)

(SEE E.G. FETTER-WALECKA, MAHAN, .)

FOR EXAMPLE FOR GREEN'S FUNCTION

$$G(X_A, X_B) \equiv -i \langle T_C \, C(X_A) \, C^+(X_B) \rangle$$

Contibution at a certain order m

1) X_A X_B

DRAW ALL TOPOLOGICALLY INEQUIVALENT CONNECTED (LINKED CLUSTER!) DIAGRAMS STARTING AT X_B AND ENDING AT X_A

with m vertices

2) ASSIGN THEIR VALUES ACCORDING TO ELEMENTS

4) MULTIPLY BY $(-1)^F$ F = N. OF FERMIONIC LOOPS

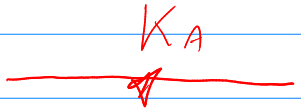
BY $(i)^{m/2}$ m = number of vertices

3) SUM OR INTEGRATE OVER ALL INTERNAL VARIABLES TIME HAS TO BE INTEGRATED OVER KELDYSH CONTOUR

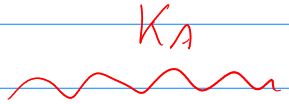
30.3

MOMENTUM SPACE MAY BE MORE CONVENIENT IN CASE OF TRANSLATION INVARIANCE

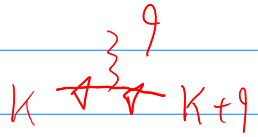
FREQUENCY SPACE FOR TIME INDEPENDENCE (EQUILIBRIUM OR STEADY STATE)



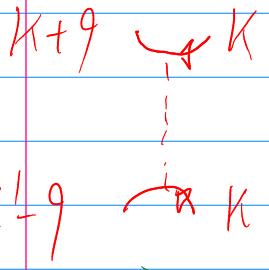
$= G_0(K_A)$



$= D_0(K_A)$



$= \mathcal{J}(K, q)$



$= \mathcal{N}(q)$

MOMENTUM CONSERVATION AT VERTICES

$= \mathcal{U}(X_A)$

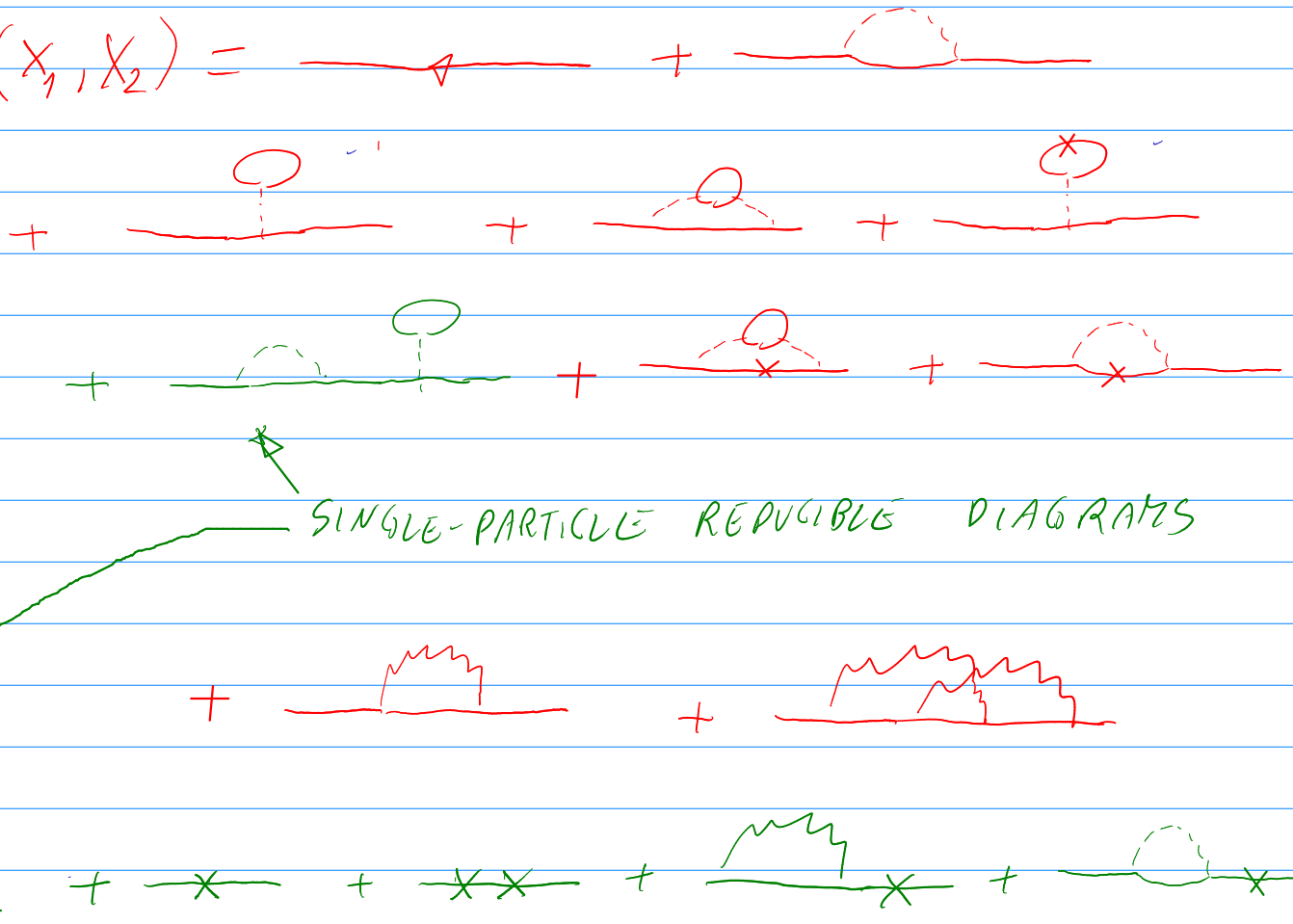
~~IN GENERAL NOT MOMENTUM CONSERVING~~

SOMETIMES MIXED REPRESENTATION

TIME + MOMENTUM OR FREQUENCY + POSITION

DYSON EQUATION

$G(X_1, X_2) =$



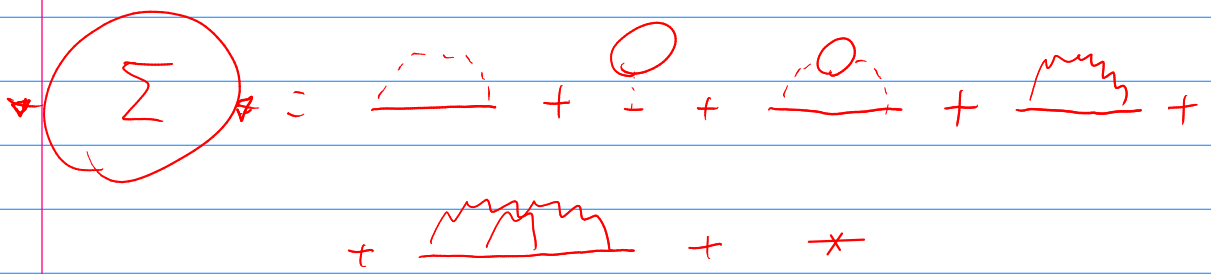
SINGLE-PARTICLE REDUCIBLE DIAGRAMS

Define self-energy Σ = sum of

single-particle irreducible diagrams

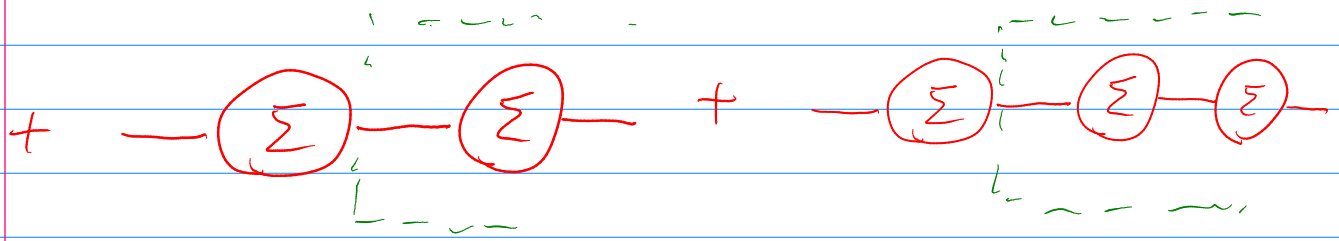
i. e. diagrams that cannot be taken apart by

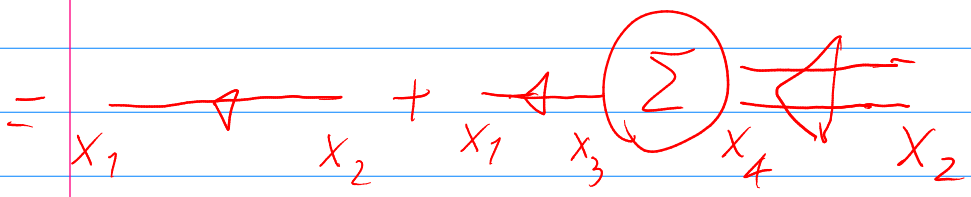
cutting a single line



$G(X_1, X_2) \equiv$ 

$=$ 

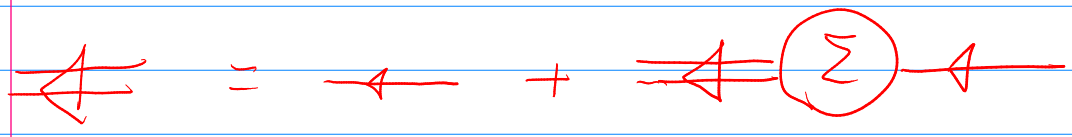
$+$ 

$=$ 

$$G(X_1, X_2) = G_0(X_1, X_2) + \int dX_3 dX_4 \, G_0(X_1, X_3) \Sigma(X_3, X_4) G(X_4, X_2)$$

INTEGRAL EQUATION FOR G

Also "left" Dyson's equation



$$G = G_0 + G * \Sigma * G_0$$

$*$ convolution, sum over internal indices

IN FREQUENCY SPACE

(EQUILIBRIUM) BECOMES EASY

$$G(W) = G_0(W) + G_0(W)\Sigma(W)G(W)$$

CAN BE SEEN AS A MATRIX EXPRESSION IN OTHER INTERNAL INDICES, POSITION/MOMENTUM, SPIN, BAND, ETC. (EVEN IN TIME)

$$G = \left(G_0^{-1} - \Sigma\right)^{-1}$$

BUT, OF COURSE, Σ IS DIFFICULT

KELDYSH SPACE

IT IS CONVENIENT TO REPLACE THE DOUBLE CONTOUR BY A MATRIX MULTIPLICATION

REPLACE (SIMILARLY FOR OTHER TWO-POINT FUNCTIONS)

$$G(\gamma_A, \gamma_B) \equiv G(t_A c_A, t_B c_B) \quad \text{WITH}$$

$$\hat{G}(t_A, t_B) = \begin{pmatrix} G^c(t_A, t_B) & G^<(t_A, t_B) \\ G^>(t_A t_B) & G^{\bar{c}}(t_A, t_B) \end{pmatrix} \qquad (A22)$$

($G(t_A c_1, t_B c_1)$)

$$G^c(t_A, t_B) \equiv -i \langle T\, C(t_A)\, C^+(t_B) \rangle$$

$$G^{\bar{c}}(t_A, t_B) \equiv -i \langle \bar{T}\, C(t_A)\, C^+(t_B) \rangle$$

$$G^<(t_A, t_B) = -i\eta \langle C^+(t_B)\, C(t_A) \rangle \qquad \text{"Lesser" GF}$$

($\eta = -1$ FERMIONS, $+1$ BOSONS)

$$G^>(t_A, t_B) = -i \langle C(t_A)\, C^+(t_B) \rangle \qquad \text{"Greater" GF}$$

CONVOLUTIONS OVER INTERNAL TIMES BECOME

$$C(\gamma_A, \gamma_B) = \int_C A(\gamma_A, \gamma)\, B(\gamma, \gamma_B)\, d\gamma =$$

$$= \int_{-\infty}^{+\infty} A(t_A C_A, t C_1)\, B(t C_1, t_B C_B)\, dt$$

$$- \int_{-\infty}^{+\infty} A(t_A C_A, t C_2)\, B(t C_2, t_B C_B)\, dt$$

FOR THE CORRESPONDING MATRICES THIS GIVES:

$$\hat{C}(t_A, t_B) = \int_{-\infty}^{+\infty} dt\ \hat{A}(t_A, t) \cdot \gamma_3 \cdot \hat{B}(t, t_B) \qquad \text{(A23)}$$

γ_3: PAULI MATRIX

THE 4 GREEN'S FUNCTIONS ARE LINEARLY DEPENDENT

$$G^c + G^{\bar{c}} = G^> + G^<$$

SO ONLY THREE ARE LINEARLY INDEPENDENT

THERE ARE SEVERAL CONVENTIONS, WE USE THE MATRIX

$$\underline{G} = L \hat{\gamma}_3 \hat{G} L^+ \qquad \text{(A24)}$$

WHERE $L = \frac{1}{\sqrt{2}}(\gamma_0 - i\gamma_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (A241)

γ_i = PAULI MATRICES

$\gamma_0 = \underline{1}$

$$\underline{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} \qquad \text{(A242)}$$

WHERE WE HAVE USED THE RELATIONS

$$G^R = G^c - G^< = G^> - G^{\bar{c}}$$

$$G^a = G^c - G^> = G^< - G^{\bar{c}}$$

$$G^K = G^> + G^< = G^c + G^{\bar{c}} \qquad \text{(A25)}$$

Consider now the convolution

$$\hat{C}(t_A, t_B) = \int_{-\infty}^{+\infty} dt \; \hat{A}(t_A, t) \cdot \gamma_3 \cdot \hat{B}(t, t_B)$$

multiply by

$$L \gamma_3 \cdots L^+$$

for example:

$$L \gamma_3 \hat{C} L^+ = \underline{C}$$

we obtain

$$\underline{\underline{C}}(t_A, t_B) = \int_{-\infty}^{+\infty} dt \; \underbrace{L \gamma_3 \hat{A}(t_A, t) L^+}_{\underline{\underline{A}}} \underbrace{L \gamma_3 B(t, t_B) L^+}_{\underline{\underline{B}}} \qquad \text{(A26)}$$

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$\|$

$$\underline{\underline{C}}(t_A, t_B) = \int_{-\infty}^{+\infty} dt \; \underline{\underline{A}}(t_A, t) \underline{\underline{B}}(t, t_B) \qquad \text{(A261)}$$

no γ_3 any more !

Noninteracting Green's functions

"building blocks" of perturbation theory

$$H_0 = \sum_p \varepsilon_p c_p^\dagger c_p$$

p can in general be a quantum number labeling an eigenstate of the single-particle Hamiltonian

For free particles p=(momentum,spin)
for electrons in a crystal:(crystal momentum, band,spin)
etc.

Evolution according to H_0 (interaction representation)

$$\frac{d c_p^\dagger}{dt} = i\left[H_0, c_p^\dagger\right] = i\varepsilon_p c_p^\dagger$$

$$c_p^\dagger(t) = e^{i\varepsilon_p t} c_p^\dagger$$

$$c_p(t) = e^{-i\varepsilon_p t} c_p$$

Retarded Green's function

$$G_0^r(p,t) = -i\,\Theta(t)\,\langle [c_p(t), c_p^+(0)]_{-\eta} \rangle$$

$$= -i\,\Theta(t)\, e^{-i\varepsilon_p t}\,\langle [c_p, c_p^+]_{-\eta} \rangle$$

$$= -i\,\Theta(t)\, e^{-i\varepsilon_p t}$$

Fourier transformation

$$G_0^r(p,\omega) = -i \int dt\; e^{i(\omega - \varepsilon_p + i\delta)t}\,\Theta(t)$$

δ = convergence factor = positive infinitesimal $\delta \to 0^+$

$$= \frac{1}{\omega - \varepsilon_p + i\delta} \qquad \text{(A1)}$$

Similarly

$$G_0^a(p,t) = i\,\Theta(-t)\, e^{-i\varepsilon_p t} = G_0^r(p,-t)^*$$

$$G_0^a(p,\omega) = \frac{1}{\omega - \varepsilon_p - i\delta} = G_0^r(p,\omega)^* \qquad \text{(A2)}$$

Greater and lesser Green's functions

$$G_0^>(p t_A, p' t_B) = \delta_{pp'} G_0^>(p, t_A - t_B)$$

$$= -i \langle c_p(t_A) c_p^+(t_B) \rangle$$

$$= -i e^{-i \varepsilon_p (t_A - t_B)} \langle c_p c_p^+ \rangle$$

$$= -i e^{-i \varepsilon_p (t_A - t_B)} \bar{n}_p$$

$$\bar{n}_p \simeq 1 + \eta n_p$$

distribution function $$n_p = \left(e^{\beta(\varepsilon_p - \mu)} - \eta \right)^{-1}$$

Fourier transform

$$G_0^>(p, w) = \int_{-\infty}^{+\infty} dt\, e^{i w t} G_0^>(p, t)$$

$$= -2\pi i \, \delta(w - \varepsilon_p) \, \bar{n}_p \qquad \text{(A3)}$$

$$G_0^<(p,t) = -i\eta \langle c_p^+(0)\, c_p(t)\rangle$$

like $G^>$ with $\bar{n}_p \to \eta n_p$

$$= -i\, e^{-i\varepsilon_p t}\, \eta n_p$$

$$\boxed{G_0^<(p,w) = -2\pi i\, \delta(w-\varepsilon_p)\, \eta n_p} \qquad \text{(A4)}$$

Keldysh Green's function

$$G_0^K(p,t) = G_p^>(t) + G_p^<(t)$$

$$= -i\, e^{-i\varepsilon_p t}\left(\bar{n}_p + \eta n_p\right)$$

$$= -i\, e^{-i\varepsilon_p t}\left(1 + 2\eta n_p\right)$$

$$G_0^K(p,w) = -2\pi i\, \delta(w-\varepsilon_p)\left(1+2\eta n_p\right) \qquad \text{(A5)}$$

For the case, as here, in which n_p is a function $n(\varepsilon_p)$ we can exploit the δ – distribution to write

$$\boxed{G_0^K(p,w) = -2\pi i\, \delta(w-\varepsilon_p)\, \gamma(w)}$$

$$\gamma(w) \equiv 1 + 2\eta\, n(w) \qquad \text{(A6)}$$

By using the property

$$\frac{1}{\omega - \varepsilon_p + i0^+} = P\frac{1}{\omega - \varepsilon_p} - i\pi\delta(\omega - \varepsilon_p)$$

We can write the relation

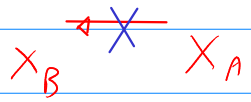
$$G_0^K(p,\omega) = \left[G_0^R(p,\omega) - G_0^A(p,\omega)\right] 2(\omega) \qquad \text{(A7)}$$

Notice that for fermions at T=0

$$2(\omega) = 1 - 2\Theta(\mu - \omega) = \text{sign}(\omega - \mu)$$

Time-dependent potential
noninteracting case

The only vertex term is



$$= \mathcal{M}(X_A, X_B)$$

let us sort out time by defining $X_i = (\gamma_i, \vec{x}_i)$

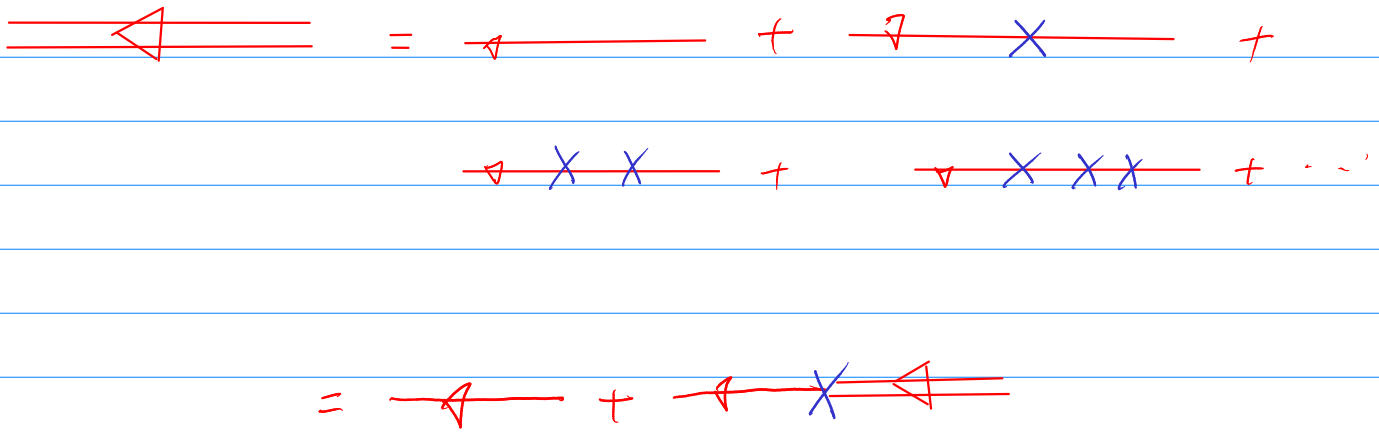
$$\gamma_i = t_i C_i$$

and replace the contour index by the 2x2 matrix representation "$\sim$"

$$\underset{\sim}{\mathcal{M}}(X_A, X_B) = \mathcal{M}(\vec{x}_A, \vec{x}_B, t_A)\,\delta(t_A - t_B) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

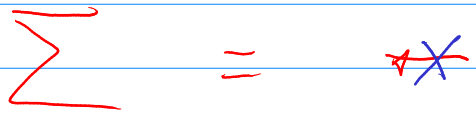
Identity in Keldysh space (does not hold in the "$\wedge$" repr.) (A8)

Diagrams



I. e. the self-energy (exactly) consists of a single

diagram: this is the only single-particle irrducible diagram



The perturbative sum can be carried out exactly

Dyson's integral equation

$$G = G_0 + G_0 * \Sigma * G$$

"$*$" is a convolution over internal variables and a matrix product in Keldysh space

$$G(x_1, x_2) = G_0(x_1, x_2) + \int dx_3 dx_4 \; G_0(x_1, x_3) \Sigma(x_3, x_4) G(x_4, x_2)$$

A simplifiction can be achieved for the "steady state"

If u is switched on at a certain time and then remains time independent, we may expect that the system reaches a stationary, i.e. time-independent state.

In this case, Green's function only depend on time difference and one can Fourier transform over time.

Some useful relations

Suppose we have to matrices in Keldysh space, i. e. of the form

$$\underline{B} = \begin{pmatrix} b_R & b_K \\ 0 & b_A \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} f_R & f_K \\ 0 & f_A \end{pmatrix}$$

We allow also the elements b_R, b_K to be matrices

$$(B \cdot F)_{ij} = B_{ik} F_{kj}$$

Then the product has the same form

$$\underline{B} \cdot \underline{F} = \begin{pmatrix} b_R f_R & b_R f_K + b_K f_A \\ 0 & b_A f_A \end{pmatrix}$$

Now, let us look at the inverse of B

If $\underline{F} = \underline{B}^{-1} \Rightarrow \underline{B} \cdot \underline{F} = I$

$$\Rightarrow f_R = b_R^{-1} \qquad f_A = b_A^{-1} \qquad \text{(A9)}$$

$$b_R f_K + b_K f_A = 0 \Rightarrow f_K = -b_R^{-1} b_K b_A^{-1} \qquad \text{(A10)}$$

(also termed Langreth Rules)