Übungsaufgaben Kap. 2

$\frac{\partial U}{\partial T}$, $\frac{\partial \langle M \rangle}{\partial h}$

2.1 (i) Verify $\langle (M - \langle M \rangle)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2 = kT\chi_T$

(ii) Show in a similar way that the fluctuations in the energy are related to the specific heat at constant volume by

$$(\Delta E)^2 \equiv \langle (E - \langle E \rangle)^2 \rangle = kT^2 C_V.$$

Use this equation to argue that $\Delta E \sim N^{1/2}$ where N is the number of particles in the system.

2.2 A paramagnetic solid contains a large number N of non-interacting, spin-1/2 particles, each of magnetic moment μ on fixed lattice sites. This substance is placed in a uniform magnetic field H.
(i) Write down an expression for the partition function of the solid, neglecting lattice vibrations, in terms of $x = \mu H/kT$.
(ii) Find the magnetization M, the susceptibility χ, and the entropy S, of the paramagnet in the field H.
(iii) Check that your expressions have sensible limiting forms for $x \gg 1$ and $x \ll 1$. Descibe the microscopic spin configuration in each of these limits.
(iv) Sketch M, χ, and S as a function of x. and of β
[Answers: (i)$Z = (2\cosh x)^N$; (ii)$M = N\mu \tanh x$, $\chi = N\mu^2/(kT\cosh^2 x)$, $S = Nk\{\ln 2 + \ln(\cosh x) - x\tanh x\}$.]

2.3 Determine the critical exponents λ for the following functions as $t \to 0$:

(i) $f(t) = At^{1/2} + Bt^{1/4} + Ct$
(ii) $f(t) = At^{-2/3}(t+B)^{2/3}$
(iii) $f(t) = At^2 e^{-t}$
(iv) $f(t) = At^2 e^{1/t}$
~~(v) $f(t) = A\ln\{\exp(1/t^4) - 1\}$~~

[Answers: (i)1/4, (ii)−2/3, (iii)2, (iv)undefined, (v)−4.]

2.4 Show that the following functions have a critical exponent $\lambda = 0$ in the limit $t \to 0$:

(i) $f(t) = A\ln|t| + B$
(ii) $f(t) = A - Bt^{1/2}$
(iii) $f(t) = 1,\ t < 0;\qquad f(t) = 2,\ t > 0$
(iv) $f(t) = A(t^2 + B^2)^{1/2}(\ln|t|)^2$
(v) $f(t) = At\ln|t| + B$

~~2.5~~[12] Consider a model equation of state that can be written

$$H \sim aM(t + bM^2)^\theta; \quad 1 < \theta < 2; \quad a, b > 0.$$

near the critical point. Find the exponents β, γ, and δ

[Answer: $\beta = 1/2$, $\gamma = \theta$, $\delta = 1 + 2\theta$.]

~~2.6~~[12] The spontaneous magnetization per spin of the spin-1/2 Ising model on the square lattice is

$$\langle s \rangle^8 = 1 - (\sinh 2J/kT)^{-4}.$$

Show that this can be written in the form

$$\langle s \rangle = B(-t)^\beta \{1 + b(-t)\ldots\}$$

where $t = (T - T_c)/T_c$ and $\beta = 1/8$. Find B and b and hence estimate the range of temperatures over which it is reasonable to ignore the correction to the leading scaling behaviour.
[Answer: $B = (8\sqrt{2}K_c)^{1/8}$, $b = (1 - 9K_c/\sqrt{2})/8$ where $K_c = J/kT_c$.]

$F = U - TS$