PROOF OF UNIVERSAL PERIMETER LAW BEHAVIOUR OF THE WEGNER–WILSON LOOP IN NONABELIAN LATTICE GAUGE THEORIES WITH HIGGS FIELDS

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By an application of reflection positivity with respect to oblique lattice planes it is proven that in lattice gauge theories with scalar matter fields the Wegner–Wilson loop is bounded from below by exp(-const.·perimeter).

For pure lattice gauge theories the qualitatively different behaviour of the Wegner–Wilson loop in different phases – area decay in the confinement phase and perimeter decay in the free charge phase – provides an important order parameter. However, if the gauge fields are coupled to dynamical matter fields, then pair creation can lead to a shielding of the gauge charges and one has to expect [1] perimeter behaviour of the Wegner–Wilson loop in all phases of the system, namely

$$W(L) := \langle \chi(U(L)) \rangle \geqslant \text{const.} \cdot \exp(-\alpha \cdot |L|) \qquad (1)$$

(here $|L|$ is the perimeter of the loop L, $U(L)$ the product of gauge fields around the loop), provided χ is the character of a representation of the gauge group under which the matter fields (or polynomials of them) transform.

In this letter we present a general proof of the inequality (1) for abelian and nonabelian lattice Higgs theories. In abelian Higgs models with a fixed length of the Higgs fields relation (1) follows from Griffith inequalities which are known to hold for such models [2]. In the non-abelian case no general proof seems to exist. Recently Borgs [3] proved the perimeter law in the strong coupling region of pure lattice gauge theories for representations which are trivial on the center of the gauge group. His methods can probably be applied also to the present problem to yield the perimeter law in the convergence region of the strong coupling expansion. In contrast to the lower bound (1) an upper bound for the perimeter behaviour was established under quite general assumptions by Simon and Yaffe [4] for pure gauge theories. If the norm $\|\Phi\|$ of the Higgs field stays bounded, e.g. $\|\Phi\| = 1$, then their result can easily be extended to Higgs systems.

Our proof of (1) relies on reflection positivity with respect to oblique lattice hyperplanes, e.g. $\Pi_+ = \{x \in \mathbb{Z}^D, x^4 = x^1\}$. In ref. [5] this property has been derived for the U(1) Higgs model, and it has been shown to imply the inequality

$$G(R) \leqslant \text{const.} \cdot W(R, R)^{1/4} . \qquad (2)$$

Here $G(R) = \langle \overline{\Phi(y)}\ U(\Gamma)\ \Phi(z) \rangle$, Φ is the Higgs field, Γ a straight lattice path of length R from y to z, and

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$W(R, S)$ the Wegner–Wilson loop (1) for a rectangular loop with side lengths R and S.

Let us first show that $G(R)$ always decays exponentially. This may be seen in the temporal gauge (we call the direction of Γ the time direction) by exploiting reflection positivity through hyperplanes perpendicular to Γ both through lattice sites and halfway between lattice sites. Such reflection positivity assures [6] the existence of the transfer matrix T, which implies that

$$G(R)=(\hat{\Phi}(y)\Omega,\, T^R\hat{\Phi}(y)\Omega)\,, \tag{3}$$

where $(\cdot,\cdot)$ denotes the quantum mechanical scalar product, Ω the vacuum and T the transfer matrix and $\hat{\Phi}(y)$ is the time-zero Higgs field operator. Since $0\leqslant T\leqslant 1$, we have the spectral representation

$$(\hat{\Phi}(y)\Omega,\, T^R\hat{\Phi}(y)\Omega)=\int \mathrm{d}\mu(\lambda)\,\lambda^R\,, \tag{4}$$

where μ is a positive bounded measure with support in the interval $[0, 1]$. Hölder's inequality yields

$$\int \mathrm{d}\mu(\lambda)\,\lambda \leqslant \left(\int \mathrm{d}\mu(\lambda)\,\lambda^R\right)^{1/R}\left(\int \mathrm{d}\mu(\lambda)\right)^{(R-1)/R}\,, \tag{5}$$

hence

$$G(R)\geqslant G(0)[G(1)/G(0)]^R\,. \tag{6}$$

Since $G(1)\neq 0$ for a nonvanishing hopping parameter, inequalities (6) and (2) together provide an alternative proof of (1) for the U(1) Higgs model.

In contrast to the proof relying on Griffith inequalities the above proof which uses reflection positivity admits a generalization to the nonabelian case as we will now show. Let G be a compact group and U an n-dimensional unitary representation of G. Consider, on a hypercubic lattice $\mathbb{Z}^D$, the following fairly general action of gauge fields coupled to scalar matter fields:

$$S_\Lambda=-\sum_{p\in \mathrm{P}_\Lambda}\chi(g_{\partial p})-\kappa\sum_{b\in \mathrm{B}_\Lambda}(\Phi(\partial_0 b),\, U(g(b))\,\Phi(\partial_1 b))+\sum_{x\in\Lambda}V(\Phi(x))\,, \tag{7}$$

where Λ is a box in $\mathbb{Z}^D$, P_Λ the set of oriented plaquettes and B_Λ the set of oriented bonds in Λ. Φ denotes an n-component complex field defined on the lattice sites and g a group valued gauge field defined on the lattice bonds b with $g(b^{-1})=g(b)^{-1}$, where b^{-1} denotes the bond b with inverse orientation. χ is an invariant function of positive type on G (i.e. a positive linear combination of characters), $g_{\partial p}$ is the usual plaquette variable, i.e. the conjugacy class of the product $\prod g(b)$, where $b\in\partial p$, with an arbitrary starting point. $\partial_0 b$ is the initial and $\partial_1 b$ the final point of the bond b. Finally V (the Higgs potential) is such that $\exp[-V(\Phi)]\,\mathrm{d}\Phi$ is a G invariant measure on C^n.

Now let θ_+ denote the reflection through the hyperplane Π_+. For each functional F of the matter and gauge field configuration (Φ, g) we define the reflection operator θ_+ by

$$(\theta_+ F)(\Phi, g)=\overline{F(\Phi^{\theta_+}, g^{\theta_+})}\,, \tag{8}$$

with $\Phi^{\theta_+}(x)=\Phi(\theta_+ x)$ and $g^{\theta_+}(b)=g(\theta_+ b)$. For Λ invariant under θ_+ we show as in ref. [5] that reflection positivity

$$\int\prod_{x\in\Lambda}\mathrm{d}\Phi(x)\prod_{b\in B_\Lambda}\mathrm{d}g(b)\,\exp[-S_\Lambda(\Phi, g)]F\theta_+(F)\geqslant 0 \tag{9}$$

holds for functionals F which depend only on fields in the half space $x^4\geqslant x^1$, $x\in\Lambda$.

Let us briefly sketch the proof. Inequality (9) becomes obvious if one replaces the action by zero. So it suffices to show that $\exp(-S_\Lambda)$ is an element of the multiplicative and additive cone

$$\mathrm{C}=\left\{ \sum F_i \theta_+(F_i) \middle| F_i \text{ localized in } x^4 \geqslant x^1 \right\} . \tag{10}$$

We split S_Λ into three pieces,

$$S_\Lambda = S_+ + \theta_+(S_+) + S_\mathrm{c} , \tag{11}$$

where S_+ is localized in $x^4 \geqslant x^1$ and

$$S_\mathrm{c} = -\sum_{p \in P_\mathrm{c}} \chi(g_{\partial_p}) , \tag{12}$$

with $\mathrm{P_c}$ denoting the set of plaquettes p which are cut by Π_+. (Terms in S_Λ with support completely inside Π_+ are unaffected by θ_+ and can thus be split equally between S_+ and $\theta(S_+)$). Now let $\chi(g)=\Sigma_\sigma \beta_\sigma \operatorname{Tr} U_\sigma(g)$, $\beta_\sigma \geqslant 0$ denote the expansion of χ into simple characters. Let $\partial p=(\partial p)_+ \cdot (\partial p)_-$ be the decomposition of ∂p into the parts above and below Π_+. We have

$$\begin{aligned} \operatorname{Tr} U_\sigma(g_{\partial_p}) &= \operatorname{Tr} U_\sigma(g(\partial_p)_+)\, U_\sigma(g(\partial_p)_-) \\ &= \sum_{i,j} U_{\sigma,ij}(g(\partial p)_+)\, U_{\sigma,ji}(g(\partial_p)_-) = \sum_{i,j} U_{\sigma,ij}(g(\partial_p)_+)\, \overline{U_{\sigma,ij}(g(\partial_p)_-^{-1})} , \end{aligned} \tag{13}$$

which is in C since $(\partial p)_-^{-1} = \theta_+((\partial p)_+)$. Since C is multiplicative and additive, also $\exp(-S_\mathrm{c}) \in \mathrm{C}$, and the same holds for $\exp(-S_\Lambda) = \exp(-S_+)\theta_+(\exp(-S_+)) \exp(-S_\mathrm{c})$. Hence reflection positivity also holds in the nonabelian case.

We conclude that there are infinite volume Gibbs states satisfying reflection positivity with respect to Π_+, if the appropriate thermodynamical limits exist. Clearly this property remains true if there is more than one Higgs field in the action (7).

Next we prove two lemmata which provide interim results for the proof of inequality (1). Here we will need reflection positivity, in order to be able to define a scalar product and use Schwartz' inequality. Assume that $\langle ... \rangle$ is an infinite volume Gibbs state which is reflection positive with respect to the hyperplanes $x^\mu = a/2$ or $x^\mu \pm x^\nu = a$, $a \in \mathbb{Z}$, $\mu \neq \nu$, μ, $\nu = 0, ..., D-1$. $\langle ... \rangle$ is automatically translation invariant. Define

$$U(\Gamma) = \prod_{b \in \Gamma} U(g(b)) \tag{14}$$

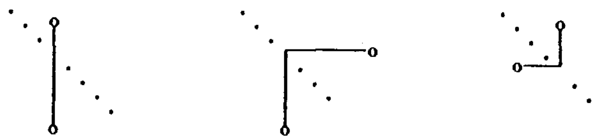
(path ordered) and

$$V(\Gamma) = (\Phi(\partial_0 \Gamma), U(\Gamma) \Phi(\partial_1 \Gamma)) \tag{15}$$

for a lattice path Γ. Let Π be one of the hyperplanes for which reflection positivity holds, and let θ be the reflection through Π and θ the associated reflection operator. We consider a path Γ from y to z consisting of two pieces $\Gamma = \Gamma_1 \circ \Gamma_2$ such that Γ_1 is the part of Γ below Π and Γ_2 the part above Π. Reflection positivity yields Lemma 1:

Lemma 1.

$$|\langle V(\Gamma_1 \circ \Gamma_2) \rangle| \leqslant \langle V(\Gamma_1 \circ \theta \Gamma_1^{-1}) \rangle^{1/2} \quad \langle V(\theta \Gamma_2^{-1} \circ \Gamma_2) \rangle^{1/2} .$$



(Here and in the following we indicate pictorially the reflections at the various hyperplanes to illustrate the steps involved.

Proof. $V(\Gamma_1 \circ \Gamma_2)$ can be written as a sum over products $\sum F_i G_i$ with F_i localized below and G_i above Π:

$$\begin{aligned} V(\Gamma_1 \circ \Gamma_2) &= (\Phi(y),\, U(\Gamma_1 \circ \Gamma_2)\Phi(z)) = (\Phi(y),\, U(\Gamma_1)U(\Gamma_2)\Phi(z)) \\ &= (U(\Gamma_1)^* \Phi(y),\, U(\Gamma_2)\Phi(z)) = (U(\Gamma_1^{-1})\Phi(y),\, U(\Gamma_2)\Phi(z)) \\ &= \sum_i \overline{(U(\Gamma_1^{-1})\Phi(y))_i}\,(U(\Gamma_2)\Phi(z))_i \equiv \sum_i F_i G_i\,, \end{aligned}$$

with $F_i = \overline{(U(\Gamma_1^{-1})\Phi(y))_i}$ and $G_i = (U(\Gamma_2)\Phi(z))_i$. In the same way one obtains $V(\Gamma_1 \circ \Gamma_1^{-1}) = \sum F_i \theta F_i$ and $V(\theta\Gamma_2^{-1}\Gamma_2) = \sum \theta(G_i) G_i$. The statement of the lemma is now simply Schwartz' inequality for the scalar product defined by reflection positivity and the sum over i. **q.e.d.**

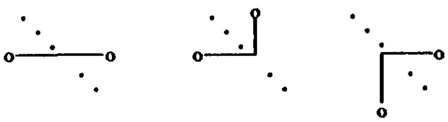
In the case where y and z lie below Π and $\Gamma = \Gamma_1 \circ \Gamma_2 \circ \Gamma_3$ with Γ_1 and Γ_3 below and Γ_2 above Π we find in an analogous way lemma 2.

Lemma 2.

$$|\langle V(\Gamma_1 \circ \Gamma_2 \circ \Gamma_3)\rangle| \leqslant \langle V(\Gamma_1 \circ \theta\Gamma_1^{-1}) V(\theta\Gamma_3^{-1} \circ \Gamma_3)\rangle^{1/2} \langle \mathrm{Tr}\, U(\theta\Gamma_2^{-1} \circ \Gamma_2)\rangle^{1/2}$$

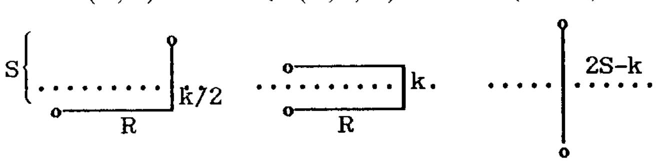
Now we turn to the proof of (1). Let $G(R_1, ..., R_k) = \langle V(\Gamma)\rangle$ where Γ is a path in a two-dimensional lattice plane consisting of straight pieces $R_1, ..., R_k$ with all angles equal to $+\pi/2$, let $W(R, S) = \langle \mathrm{Tr}\ U(\Gamma)\rangle$ for a rectangular loop with side lengths R and S and let $H(R, S) = \langle V(\Gamma)\ V(\Gamma^{-1} + x)\rangle$ where Γ is a straight path of length R and x is a translation by S in a coordinate direction orthogonal to Γ. Using lemma 1 we find the following relations:

$$G(2R) \quad \leqslant G(R, R)^{1/2} \quad G(R, R)^{1/2} = G(R, R)\,. \tag{16}$$



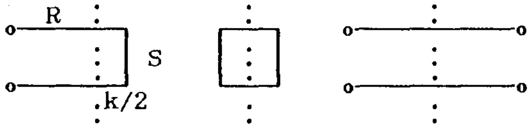
This is obtained by reflection through an oblique lattice hyperplane through the midpoint of Γ and by observing that the two resulting factors on the RHS are equal because of reflection symmetry of the system. Also:

$$G(R, S) \qquad \leqslant G(R, k, R)^{1/2} \qquad G(2S-k)^{1/2}, \qquad k \leqslant 2S\,. \tag{17}$$



Here one reflects through the $(x^\mu = k/2)$ hyperplane where the second straight piece of Γ points into the positive μ-direction and the corner is at the origin. In a similar way lemma 2 leads to

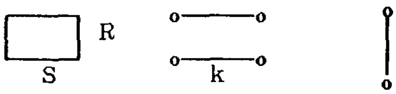
$$G(R,S,R) \leqslant W(k,S)^{1/2} \; H(2R-k,S)^{1/2}, \quad k \leqslant 2R . \tag{18}$$



Here Γ is in the μ–ν plane, the second part of Γ is in the $(x^\mu=0)$ hyperplane, and the first part points into the negative μ direction. The reflection is again through the $(x^\mu=k/2)$ hyperplane. Putting inequalities (16)–(18) together we find, for $S+k$ even, $S \geqslant R$:

$$G(S+k) \leqslant G(\tfrac{1}{2}(S+k), \tfrac{1}{2}(S+k)) \leqslant G(\tfrac{1}{2}(S+k), R, \tfrac{1}{2}(S+k))^{1/2} \; G(S+k-R)^{1/2}$$

$$\leqslant W(S,R)^{1/4} \quad H(k,R)^{1/4} \quad G(S+k-R)^{1/2} . \tag{19}$$



We now exploit the fact that as in the abelian case

$$G(R) = \int d\mu(\lambda) \; \lambda^R , \tag{20}$$

with a bounded measure μ on [0,1]. Hölder's inequality, taking as measure $\lambda^k \, d\mu(\lambda)$, yields

$$G(S-R+k) \; G(R+k) \leqslant G(S+k) \; G(k) \tag{21}$$

for all $k \geqslant 0$. Using (19), we arrive at the inequality

$$G(S+k)^2 \; G(R+k)^2 \leqslant G(k)^2 \; H(k,R) \, W(S,R) . \tag{22}$$

Inequalities of the type (16)–(18) and (22) are useful in a variety of contexts. In particular, inequality (22) implies the perimeter law for the Wegner–Wilson loop, provided supp $\mu \neq \{0\}$, i.e. there are components with finite energy in the state $\hat{\Phi}_i(y)\Omega$ generated by the application of $\hat{\Phi}$ to the vacuum. (For bounded Higgs fields it was shown in ref. [7] that there are no infinite energy states for nonvanishing hopping parameter κ.) In this case $G(k) \neq 0$ for all k and

$$G(RS+k)/G(k) \geqslant [G(1+k)/G(k)]^R , \tag{23}$$

by Hölder's inequality. Therefore from inequality (22)

$$W(S,R) \geqslant c_k(R) \exp[-\alpha_k \cdot 2(S+R)] , \tag{24}$$

with $c_k(R) = G(k)^2/H(k,R)$ and $\alpha_k = -\ln[G(k+1)/G(k)]$. Since $H(k,R)$ can be written as $(\hat{V}(\Gamma_k)\Omega, T^R \hat{V}(\Gamma_k)\Omega)$, where Γ_k is a straight path of length k, it is monotonically decreasing for $R \to \infty$ (because $0 \leqslant T \leqslant 1$) and bounded from below $(\hat{V}(\Gamma_k)\Omega, \Omega) \cdot (\Omega, \hat{V}(\Gamma_k)\Omega) = G(k)^2$. Therefore the parameters $c_k(R)$ satisfy the bound $c_k(1) \leqslant c_k(R) \leqslant 1$. Thus we have now proved inequality (1).

Since $\alpha_k \to \mu$ for $k \to \infty$, where μ is the smallest value occurring in the energy spectrum of $\hat{\Phi}_i(y)\Omega$ (or, phrased differently, μ is the lowest energy of dynamical fields in the presence of a screened external charge), we find the asymptotic relation

$$\limsup_{R,S\to\infty} \frac{1}{2(R+S)}(-\ln W(R,S)) \leqslant \mu \,. \tag{25}$$

If we denote the limit on the left side of inequality (25) by E_q and recall the relationship between the Wegner–Wilson loop and the potential $V(R)$ between external sources, we see that $E_q = \frac{1}{2}V(\infty)$ is the field energy of an external charge, for which $\mu - E_q \geqslant 0$. The latter relation was derived in ref. [5] for the abelian case, where $\mu - E_q$ was found to be a very sensitive order parameter, distinguishing between a free charge phase ($\mu - E_q > 0$) and a confinement-Higgs phase ($\mu - E_q = 0$).

The method above can be generalized to the Wegner–Wilson loop in other representations of the gauge group and by replacing the Higgs field in eq. (15) by fields which are functions of the Higgs field and the gauge field and transform under the appropriate representation of the gauge group. This includes, for example, the case of the Wegner–Wilson loop in the adjoint representation for pure gauge theories. The general result is formulated in the following theorem:

Theorem. Let the Higgs system be defined by an action of the form (7). Let σ be an irreducible representation of the gauge group and $\mu_\sigma < \infty$ the lowest energy in the sector with a screened external charge of type σ at some point. Then the associated Wegner–Wilson loop W obeys the perimeter law (inequality (24)) and in addition

$$\limsup_{R,S\to\infty} \frac{1}{2(R+S)}(-\ln W(R,S)) \leqslant \mu_\sigma \,.$$

In order to extend the above theorem to the case of fermionic matter fields, it remains to prove reflection positivity for the oblique hyperplane Π_+. All other arguments are identical.

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