Search for critical points in the SU(2) Higgs model

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We study the order of the Higgs phase transition in the SU(2) Higgs model at several values of the gauge coupling β for bare quartic coupling $\lambda=0.5$ using Monte Carlo simulations. We determine the internal energy of metastable states on various lattice sizes and estimate that the transition terminates at the critical point located at $1.95<\beta_c<2.25$. We also apply a recently proposed multihistogram method in our data analysis. At $\beta=2.25$ and $\beta=2.4$ our data strongly indicate the presence of a first-order phase transition. For $\beta\geq 2.6$ we find no indication of a discontinuity in the internal energy on lattices up to size 16^4 and the transition is either weakly first order or of higher order.

I. INTRODUCTION

The Glashow-Weinberg-Salam (GWS) gauge theory of electroweak interactions assumes the presence of the Higgs mechanism for its successful description of many phenomena. The latter is responsible for the generation of vector-boson masses, due to the coupling of the gauge fields to the self-interacting scalar fields. The Higgs sector of the theory (coupled gauge and scalar fields without fermions) is, however, not yet completely understood. One important open question is—Does a nontrivial field theory exist in the infinite-cutoff limit? Also, the nonperturbative effects of a phenomenologically allowed strong self-coupling of the scalar field should be studied. Therefore various Higgs models have been investigated in recent years by means of nonperturbative Monte Carlo simulations on the lattice. For a review and a list of references on lattice Higgs models see Ref. 1. This approach requires understanding the continuum limit of the lattice Higgs models; i.e., one has to find a suitable critical point. At such a point the system flows to a fixed point in the sense of the renormalization group and the limit of vanishing lattice constant can be performed.

One possible scenario is that the continuum limit relevant for the GWS theory is approached in the vicinity of the Gaussian fixed point of the scalar theory. It has been demonstrated then that the inclusion of the gauge fields cannot change the noninteracting character of the resulting continuum theory.[2] An interacting Higgs sector is possible only if one introduces a finite cutoff, and the GWS theory is then an effective field theory valid only at energies lower than the intrinsic scale parameter. This approach has recently led to the determination of an upper bound for the Higgs-boson mass.[3]

Although this scenario is at present acceptable both from the theoretical and phenomenological points of view, it is also important to examine the Higgs models for nontrivial fixed points where a genuine continuum limit with interacting fields could be possible. It is now nearly certain that the pure scalar field theory in four dimensions does not have such a nontrivial fixed point, but we cannot exclude the possibility that the coupled system of gauge and scalar fields behaves differently. Higgs models have large manifolds of phase transitions in the space of couplings and the fixed-point structure has not yet been reliably determined.

Motivated by these considerations we have investigated the nature of the phase transition in the SU(2) Higgs model with scalar fields in the fundamental representation. This model corresponds to the Higgs sector of the GWS theory if the U(1) gauge coupling is neglected. The model is known to have a two-dimensional sheet of "Higgs" phase-transition (PT) points in the three-dimensional space of coupling parameters β, κ, and λ (here $\beta=4/g^2$ is related to the gauge field coupling g, λ is the bare quartic self-coupling, and κ is the hopping parameter of the scalar field). For small $\lambda \leq 0.03$ and for $\beta=1.0-3.0$ the Higgs PT is distinctly of first order[4] as suggested by perturbative calculations.[5] When λ or β increase, the Higgs PT weakens and for the pure scalar field theory at $\beta=\infty$ the Higgs PT is of second order. It is possible, however, that the order of the Higgs PT at a fixed value of λ changes at some finite β thus implying the existence of a tricritical line separating a two-dimensional manifold of critical points from the phase transitions of first order. For theoretical hints in this direction see Ref. 6. In the vicinity of a possible tricritical line one could then hope to construct a nontrivial continuum limit.

From earlier numerical data[4] we concluded that $\lambda=0.5$ is a good candidate for a search for such an effect. At this λ the first order of the Higgs PT is still observable for $\beta=2.25$ but the Higgs PT is weak enough that one can expect a possible change of the order with increasing but still moderate values of β. Therefore we have fixed λ to this value throughout the paper.

In addition, for large values of λ the Higgs PT sheet is bounded by a line of critical points at small positive values of β. Thus a nontrivial continuum limit might also be possible there. Although its relevance for the GWS theory is not obvious, such a continuum limit might be of fundamental field-theoretical interest. The localization of these end points is also interesting from the point of view of recently performed investigations of the SU(2) Higgs model coupled to fermions.[7] There it is desirable to compare the positions of the Higgs and the chiral phase transitions in order to find out whether they coincide.

The scenarios for continuum limits just described might be unlikely. However, their possible impact on the GWS theory nevertheless justifies their careful examination. We face very difficult problems when using the standard Monte Carlo simulation for this purpose because it is very difficult to distinguish weakly first-order phase transitions from second-order ones on a finite lattice. Previous studies have encountered such problems in the localization of critical points and/or tricritical points in spin systems[8] and lattice gauge theories.[9] At present we hope to gain better knowledge of the various properties of the Higgs PT and to localize regions in the coupling space where further investigations, e.g., using the Monte Carlo renormalization-group (MCRG) technique would be most promising.

II. BACKGROUND

A. The model

The lattice regularized action for the SU(2) Higgs model is given by the expression

$$S=-\frac{\beta}{4}\sum_P \mathrm{tr}(U_P+U_P^\dagger) -\kappa\sum_x\sum_{\mu=1}^{4}\mathrm{Re}(\mathrm{tr}\Phi_x^\dagger U_{x,\mu}\Phi_{x+\mu}) +\lambda\sum_x \tfrac{1}{2}\mathrm{tr}(\Phi_x^\dagger\Phi_x-1)^2+\sum_x \mathrm{tr}\Phi_x^\dagger\Phi_x \ . \quad (1)$$

Here $U_{x,\mu}$ are link variables of the gauge field in the fundamental representation of the SU(2) gauge group and U_P are their products along the lattice plaquettes. The site variables of the scalar field Φ_x are written in the form $\Phi_x=\rho_x\sigma_x$ where $\sigma_x\in$SU(2) is in the fundamental representation, too, and ρ_x is the length of the scalar field. The first term describes the pure gauge field interaction; the second one the interaction between the scalar and the gauge fields and the last one the self-interaction of the scalar fields. The model has the coupling constants $\beta=4/g^2$ with g being the gauge coupling constant, κ the hopping parameter, and λ the bare quartic self-coupling, which is fixed to the value 0.5 throughout this paper. The system has only one phase: the confinement-Higgs phase. There is a Higgs PT line separating the confinement region below the phase transition (smaller values of κ) from the region above, which is the Higgs region. The line extends from $\beta=\infty$ [where the model reduces to the pure O(4)-symmetric Φ^4 theory] to small values of β, but has a critical end point at some finite β (Refs. 4 and 10).

B. Details of simulation

The simulations were carried out on Cyber 205 vector computers using the Metropolis algorithm on L^4 hypercubic lattices with periodic boundary conditions. While most of our runs were performed with a one-link update, we also performed runs with three hits for larger values of β ($\beta\geq 2.6$) thus achieving an increased acceptance rate. In order to minimize the working space requirements we simulated the icosahedral finite subgroup of SU(2) which is a very good approximation to SU(2) for the values of the couplings we have considered.[4] A fully vectorized code for the updating using a checkerboard decomposition of L^4 lattices, was implemented. We used two different random-number generators: one based on the congruent modulus method[11] and second a shift register random-number generator. No systematic difference in our results was observed. Typically we discarded 10^4 sweeps at the beginning, allowing for equilibration. Relatively far (in the κ direction) from the phase transition 10^4 sweeps were kept for measurements; close to the Higgs PT we performed up to 10^5 sweeps even on our largest lattice. We considered lattices ranging in size from 4^4 to 16^4 and performed our simulation at many values of the couplings using a total of about 10^3 h supercomputer time.

C. Finite-size effects

Phase transitions are extensively studied in the framework of statistical mechanics. The properties of the internal energy determine the nature of the phase transi-

tions. The internal energy develops a gap at a first-order phase transition in the infinite-volume limit, while at a second-order phase transition the internal energy is continuous, but its derivative with respect to its couplings becomes infinite. Performing numerical simulations on a *finite* lattice poses the problem of extrapolating finite-volume data in order to determine infinite-volume properties of the internal energy.

In our model we concentrated on the hopping term, which is the interaction term of the scalar and gauge fields

$$E=\frac{1}{8}\frac{1}{L^4}\sum_x\sum_{\mu=1}^{4}\mathrm{Re}(\mathrm{tr}\Phi_x^\dagger U_{x,\mu}\Phi_{x+\mu})\ . \tag{2}$$

Its expectation value $\langle E\rangle$ is the part of the internal energy contained in the hopping term of the action. This choice is motivated by our earlier[4] observation, that the operator (2) most clearly exhibits the critical behavior at the Higgs PT.

Various methods can be devised in order to study phase transitions in different situations. At a *strong* first-order phase transition one can study the appearance of long-living *metastable states* in the computer time sequence of generated E values E_i, which manifest themselves as double-peak structures in the distribution function $N(E)$ of E_i. It is straightforward to estimate the gap in the internal energy

$$\Delta E=E_+-E_- \tag{3}$$

by locating the maxima E_+ and E_- in the distribution function $N(E)$ on each lattice and at values of couplings where the two metastable states are equally probable. For a strong first-order phase transition one finds a gap which is insensitive to a variation of the lattice size, making the determination of the first-order nature of the phase transition with numerical methods relatively easy. We emphasize that on a finite lattice the internal energy $\langle E\rangle$ is a continuous function of the coupling constants although metastable states are present in the system. This becomes apparent numerically when the statistics of the simulation is high enough (rounding effect). It is only in the infinite-volume limit, when the Monte Carlo lifetime of the metastable states becomes infinite, that the internal energy develops a discontinuity.

The situation, however, becomes much more complicated if the transition is either of *weak* first order or of second order, but close to a first-order line. Here the locations of possible double peaks E_+ and E_- and the gap size ΔE become functions of the lattice size and even a large gap on a small lattice can decrease to a small value if one increases the lattice size. It is then naturally very hard to decide from a numerical simulation what its value is in the infinite-volume limit. It may even happen that a double-peak distribution function $N(E)$ on a finite lattice approaches a Gaussian form in the thermodynamic limit and the phase transition is actually of second order. Such phenomena can be observed in the vicinity of tricritical points and critical points.

In our simulation we monitored the computer time evolution of values E_i calculated in single configurations separated by 10 sweeps. Typically the statistics at the phase transition points was large enough to allow a few phase flips or fluctuations of the system, even on larger lattices. We then determined the distribution function $N(E)$ by binning the obtained values of E_i into 50 bins and determined E_+ and E_- from the location of the maxima, if they were present.

We also tried to refine this approach by replacing the sequence of values E_i by a sequence of smoothed values E_j^*. We calculated the averages of E_i values in blocks with length N_S and with the center at j,

$$E_j^*=\frac{1}{N_S+1}\sum_{|j-i|\le N_S/2}E_i\ , \tag{4}$$

and determined the corresponding distribution functions $N^*(E^*)$. This approach is motivated by the idea that one should be able to eliminate fluctuations on short scales in computer time but still preserve the dynamics of the phase flips. This is especially useful if the fluctuations in each metastable state are not much smaller than the gap size itself. The distribution function $N^*(E^*)$ of the smoothed E values can then reveal the locations E_+ and E_- of the internal energy maxima in a much better way than $N(E)$.

The second moment of the distribution function—the specific heat—measures the fluctuations of the internal energy around its average value

$$C=\partial_\kappa\langle E\rangle=8L^4\langle(E-\langle E\rangle)^2\rangle\ . \tag{5}$$

Its finite-size scaling analysis has been the standard tool for the study of second-order phase transitions.[12] In particular one can determine C as a function of κ (κ plays the role of the inverse temperature for fixed values of λ and β) and then study its maximal values $C_{\max}$ as a function of the lattice size L. The divergence of the specific heat with L is described by the scaling law

$$C_{\max}\propto L^{\alpha/\nu} \tag{6}$$

for very large lattices with critical exponents α and ν. We note that $C_{\max}$ diverges even at a first-order phase transition: the fluctuations between the metastable states dominate the fluctuations of the internal energy and the specific heat diverges with the volume of the system,[13] i.e., $C_{\max}\propto L^4$ in our case. We determined the specific heat by means of the second expression in Eq. (5) and its error was determined by blocking the data for C into few blocks of typical length 1000–5000 sweeps. Those entries were then treated as independent measurements.

We also considered the modified fourth cumulant V_L defined by

$$V_L=1-\frac{\langle E^4\rangle_L}{3\langle E^2\rangle_L^2}\ . \tag{7}$$

This quantity behaves quite differently at first- and second-order phase transitions. At a second-order phase transition its value in the thermodynamic limit is $\frac{2}{3}$ (reflecting the fact that the corresponding distribution functions approach Gaussian forms), this value is more-

over independent of the temperature. For a first-order phase transition, however, V_L will have a value *smaller* than $\frac{2}{3}$ on the phase transition point in the thermodynamic limit. This property and the finite-size dependence of V_L has been successfully used in statistical mechanics models to study the order of phase transitions.[13,14]

D. Multihistogram method

It is very difficult in numerical simulations to reliably determine, e.g., the maximum of the specific heat $C_{\max}$ on a given lattice because the precise location of the PT is usually not known. While one can obtain a reasonable estimate by performing a large number of MC runs in the close vicinity of the PT (in our model we chose a set of κ_γ with $\gamma=1,\ldots,N$ at fixed β), the problem of principle remains: it is most likely that the MC simulation misses the finite-volume phase-transition point.

Recently Ferrenberg and Swendsen (FS) proposed a method which allows an analytic continuation of thermodynamic quantities, e.g., the specific heat, in the critical region of the model.[15] Making use of the analytic form of the partition function, and combining the measured distribution functions $N_\gamma(E)$ as determined in the MC simulation at couplings κ_γ, the probability distribution function $P(E,\kappa)$ is determined by means of the equation

$$P(E,\kappa)=\frac{\sum_{\gamma=1}^{N} N_\gamma(E)\exp(8\kappa L^4 E)}{\sum_{\gamma=1}^{N} n_\gamma \exp(8\kappa_\gamma L^4 E - F_\gamma)} . \tag{8}$$

Here is n_γ the total number of entries into the (unnormalized) histogram at κ_γ and independent MC measurements are assumed. The set of free energies F_γ with $\gamma=1,\ldots,N$ can be determined self-consistently by Eq. (8) and

$$\exp(F_\gamma)=\sum_E P(E,\kappa_\gamma) . \tag{9}$$

The specific heat can then be calculated by means of

$$C(\kappa)=8L^4\sum_E P(E,\kappa)(E^2-\langle E\rangle^2) . \tag{10}$$

Analogous formulas hold for the mean of the internal energy $\langle E\rangle$ and the modified fourth cumulant V_L. An estimate of the errors can be obtained by applying the multihistogram method for different subsets of κ_γ values.

III. RESULTS AND DISCUSSION

We studied the model (1) for $\lambda=0.5$ at values $\beta=1.95$, 2.1, 2.25, 2.4, 2.6, and 3.5 varying the hopping parameter κ. Figure 1 shows $\langle E\rangle$ on the 16^4 lattice in the vicinity of the Higgs PT. In the cases of $\beta=2.1$ and 2.25 we observe a rapid variation of $\langle E\rangle$ as we cross the phase transition whereas for the larger β values and also for $\beta=1.95$ the curves are much smoother. We expect that at $\beta=2.4$, 2.6, and 3.5, in spite of this smoothness, the Higgs PT is present, as this transition is believed to continue to $\beta=\infty$. At $\beta=1.95$, however, the data indicate

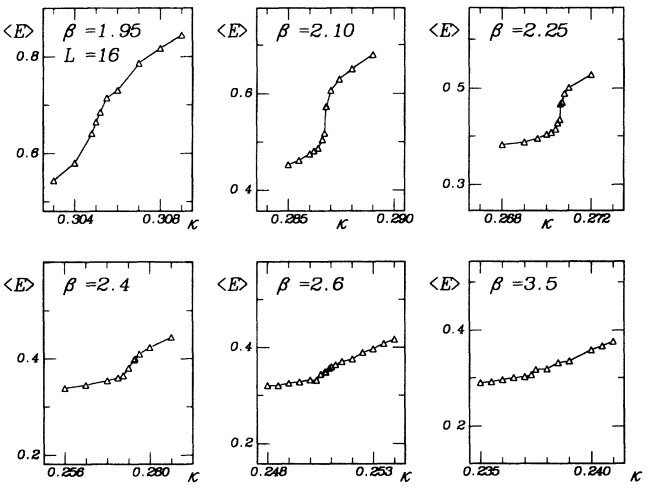

FIG. 1. The internal energy $\langle E\rangle$ as a function of κ on 16^4 lattices for various values of β.

the absence of a phase transition; one can only see a rounded, nondivergent specific-heat peak at $\kappa=0.305(1)$.

We determined the positions of the Higgs PT on the 16^4 lattice from the distribution functions $N(E)$. At $\beta=2.1$, 2.25, and 2.4 double peaks are present and it is possible to determine the metastability region. Its width allows us to estimate the error of the position of the Higgs PT points. At $\beta=2.6$ and 3.5 we did not observe metastability on our lattices and we therefore determined the location of the phase transition from the location of the peak of the specific heat. The phase-transition points, determined on the 16^4 lattice, are listed in Table I. In Fig. 2 we draw the corresponding phase-transition line in the β-κ plane.

It is already clear from Fig. 1 that the Higgs PT at

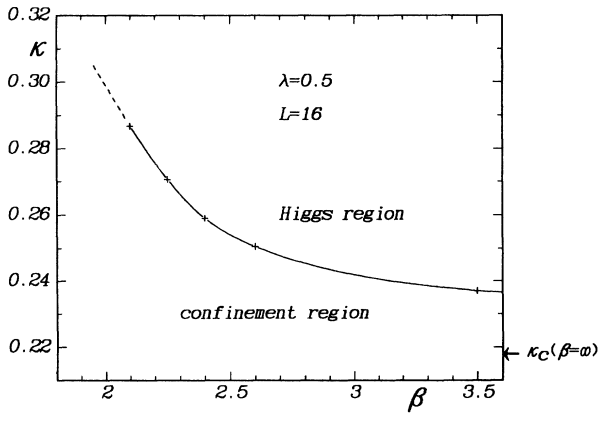

FIG. 2. The phase diagram of the SU(2) Higgs model at $\lambda=0.5$ on 16^4 lattices. The positions of the phase-transition points are listed in Table I. We indicate the transition point at $\beta=\infty$ [scalar O(4) theory]. The solid line ends at the most probable position of the critical point and the dashed line reflects the possible continuation allowed by the errors.

TABLE I. Positions of the Higgs phase-transition points determined on the 16^4 lattice. At $\beta=1.95$ the position of the specific-heat peak is indicated by an asterisk.

β	κ_{PT}
1.95	0.305(1)*
2.1	0.286 75(10)
2.25	0.270 65(5)
2.4	0.259 00(20)
2.6	0.250 60(30)
3.5	0.236 50(50)

$\lambda=0.5$ is not a strong first-order phase transition anywhere. At some of the considered β values we nevertheless detected metastability phenomena with small gaps ΔE depending on the lattice size. Here we present an analysis of these gaps for various β and lattice sizes. The results are collected in Table II.

For $\beta=2.25$ we display in Fig. 3 a sequence of distribution functions $N(E)$ obtained on lattices $L=6$, 8, 12, and 16. The κ values are always chosen such that the double-peak structure in $N(E)$ is approximately symmetric. On larger lattices a clear double-peak structure due to metastable states is seen, becoming more pronounced as the lattice size increases. For the 6^4 lattice the distribution $N(E)$ is dome shaped without pronounced peaks. Here we tried the smoothing technique which was described in Sec. II C. The utility of this approach is demonstrated in Fig. 4 where we display the distribution $N^*(E^*)$ of the smoothed values E_j^* for smoothing length $N_S=50$. Double peaks become clearly visible. We determined E_+ and E_- from the positions of the maxima of the cubic spline interpolation curves, also

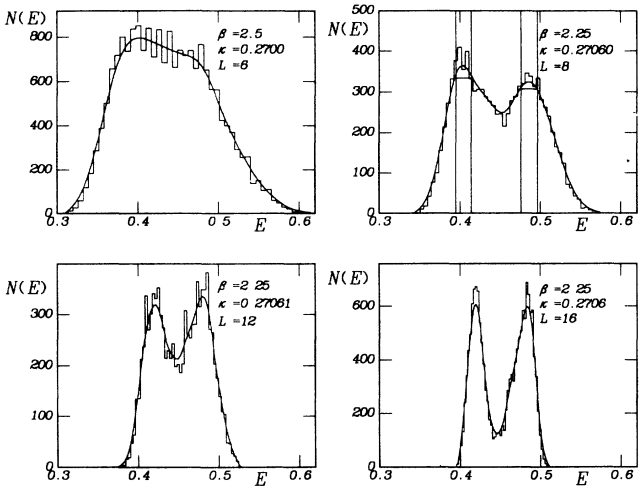

FIG. 3. A sequence of distribution functions $N(E)$ at $\beta=2.25$ on lattices of size $L=6$, 8, 12, and 16. The curves are cubic spline interpolations. The vertical solid lines plotted for the 8^4 lattice illustrate the error estimates of E_+ and E_-.

shown in Figs. 3 and 4. The errors were estimated as follows. For runs with very high statistics the data were divided into several computer time bins and E_+ and E_- were determined for each bin from the position of the maxima of the spline interpolation functions to $N(E)$ or $N^*(E^*)$. The errors were calculated from the scattering of the positions of the maxima. We found that the error corresponds approximately to the width of the peaks as 95% of the maximal height. This error estimate was adopted also for runs with a smaller statistics, when a bin

TABLE II. Positions E_+ and E_- of metastable states and ΔE depending on β, κ, and L. Values of the gap marked with an asterisk have been obtained with the smoothing technique.

β	L	κ	E_-	E_+	$\Delta E=E_+-E_-$
2.1	4	0.283 50	0.420±0.015	0.580±0.018	0.160±0.033*
		0.284 00	0.421±0.014	0.593±0.016	0.172±0.030*
	6	0.286 50	0.461±0.011	0.604±0.021	0.143±0.032*
		0.287 00	0.480±0.012	0.620±0.012	0.140±0.024*
	8	0.286 50	0.476±0.008	0.585±0.012	0.109±0.020*
		0.286 80	0.505±0.009	0.608±0.010	0.103±0.019*
	12	0.286 70	0.509±0.009	0.590±0.007	0.081±0.016
		0.286 80	0.501±0.008	0.579±0.008	0.078±0.016
	16	0.286 77	0.535±0.010	0.590±0.005	0.055±0.015
		0.286 80	0.535±0.010	0.595±0.005	0.060±0.015
2.25	4	0.270 00	0.377±0.010	0.479±0.008	0.102±0.018*
	6	0.270 00	0.384±0.006	0.466±0.008	0.082±0.014*
	8	0.270 60	0.405±0.010	0.487±0.011	0.082±0.021
		0.270 80	0.406±0.004	0.496±0.007	0.090±0.011
	12	0.270 55	0.418±0.005	0.476±0.006	0.058±0.011
		0.270 61	0.421±0.006	0.478±0.006	0.057±0.012
	16	0.270 60	0.419±0.003	0.483±0.004	0.064±0.007
2.4	8	0.258 70	0.367±0.007	0.397±0.005	0.030±0.012*
	16	0.259 00	0.366±0.002	0.393±0.002	0.027±0.004

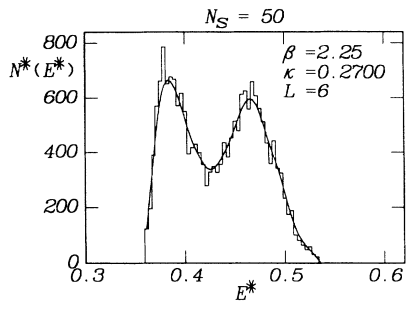

FIG. 4. Distribution $N^*(E^*)$ of the smoothed values E^*. The smoothing length is 50 sweeps.

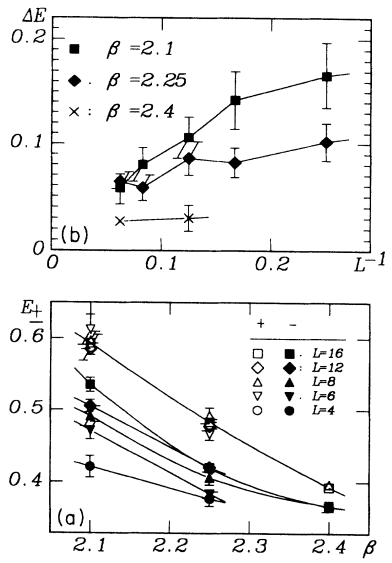

FIG. 5. (a) The gap ΔE as a function of L^{-1} for several β. (b) Metastable states E_+ and E_-. The lines are cubic spline interpolations of E_- for all lattice sizes, and of E_+ (in order to maintain clarity) only for the 16^4 lattice.

analysis was not possible (see Fig. 3 for an example of this procedure). We remark that for different κ the values of E_+ and E_- are consistent within the errors, as long as both peaks are seen (see Table II), they are also nearly independent of a change of the smoothing length N_S. For some of our distribution functions we also used double Gaussians and found values of E_+ and E_- consistent with the determination described above.

For $\beta=2.1$ and $L \leq 12$ the situation is analogous to $\beta=2.25$ and using distribution functions and the smoothing technique we again found metastable states on lattices of size $L=4$, 6, 8, and 12 and determined their positions. On the 16^4 lattice we did not find clear double peaks in histograms even when utilizing the smoothing technique. The metastable states are only indicated by two shoulders in $N(E)$, whose positions (the values of E_+ and E_-) can be determined quite reliably, however. Inspecting the evolution of E_i with computer time we found long and smooth fluctuations; critical slowing down behavior of the system was apparent.

The situation for $\beta=2.4$ is comparable to that at $\beta=2.25$. We considered only 8^4 and 16^4 lattices and observed clear metastable behavior of the system in both cases. The gaps on both lattices are, within the errors, of the same size. We note that at $\beta=2.4$ the lifetime of the metastable states is even larger than at $\beta=2.25$ for the same lattice sizes, in spite of a smaller gap at $\beta=2.4$.

At $\beta=1.95$, 2.6, and 3.5 we did not find signals for metastability even when using the smoothing technique on the 12^4 and 16^4 lattices. The distribution functions sampled appear to be close to Gaussian. Of course, this does not exclude the possible existence of slightly separated metastable states on even larger lattices.

In Fig. 5(a) we summarize our results for the gap ΔE at $\beta=2.1$, 2.25, and 2.4 on lattices of various sizes. At $\beta=2.1$ we find a rapid decrease of the gap size with increasing lattice size L. The data are not good enough to perform a quantitative extrapolation of the gap to $L=\infty$, but the impression is that the gap probably vanishes in the thermodynamic limit. At $\beta=2.25$ we find a slowly decreasing gap size as we increase L. The gap stabilizes, however, for the largest lattice sizes considered. Here we conclude that the gap ΔE will probably survive in the thermodynamic limit. One can conclude this also from the distinct double-peak form of the distribution functions on the 12^4 and 16^4 lattices, Fig. 3. For $\beta=2.4$ the gap size seems to be rather independent of the lattice size and we conclude again that the phase transition is probably of first order. In Fig. 5(b) we plot the internal energies E_+ and E_- of the metastable states as functions of β for various lattice sizes L. First we note that the location of the upper state E_+, which is associated with the Higgs region, is rather independent of the lattice size. The finite-size dependence of the gap size is mainly caused by the finite-size dependence of the lower state E_- associated with the confinement region. The curvature of E_- on the 16^4 lattice as a function of β is also remarkable, whereas the E_+ values lie on a straight line. We finally note that on the 16^4 lattice the maximal gap size is at $\beta \approx 2.25$. An interpolation of Fig. 5(b) of E_+ and E_- on the 16^4 lattice using cubic splines indicates that the gap for the 16^4 lattice vanishes close to $\beta \approx 2.0$. The figure also suggests that the β value at which the gap vanishes increases with increasing lattice size.

What can we say about the position of the critical point? With due caution with respect to the uncertainties associated with an interpolation to the thermodynamic limit, we conclude from the absence of metastable states at $\beta=1.95$ and from the indicated vanishing of the gap on the 16^4 lattice around $\beta \approx 2.0$ (or higher), that the critical point lies above $\beta=1.95$. On the other hand, at $\beta=2.25$ the gap seems to persist for large L; therefore, we expect the end point to lie below this β value. As seen in Table I the crossover at $\beta=1.95$ and the Higgs PT points at $\beta=2.1$ and 2.25 on the 16^4 lattice lie on an approximately straight line. Thus we expect that also the critical point lies on this line, i.e., between the points $(\beta,\kappa)=(1.95,0.305(1))$ and $(2.25,0.271(1))$. (The difference in κ_{PT} at $\beta=2.25$ and the increase of the error with respect to the values given in Table I reflects a possi-

ble shift of κ_{PT} in the thermodynamic limit.) This localization of the critical point is considerably more precise than in earlier estimates,[4] which suggests its position to be at some $\beta < 1.5$. Similar change of the position of the critical point should be expected also for other λ, in particular for $\lambda = \infty$.

The other question is—Is there a tricritical point at some finite value of β? Our data in Fig. 5(b) show a rapid decrease of the gap size with β between $\beta=2.25$ and $\beta=2.4$, and one may be tempted to estimate the location of a possible tricritical point at values of β between 2.5 and 2.7. This reasoning however assumes the existence of a tricritical point at some finite $\beta=\beta_{TCP}$ and a decrease of the gap size as some power of $\beta_{TCP}-\beta$. This might not be true and a nonvanishing but exponentially small gap ($\sim e^{-\beta}$) might persist up to the largest values of β (as indicated by perturbation theory[5]). As we cannot rule out this possibility, our claim is that if there is a tricritical point at some finite coupling, then its most probable position is in the region $\beta=2.5-2.7$. One could perform a search using, e.g., the Monte Carlo renormalization-group method. We note that for the SU(2) Higgs model with the scalar fields in the adjoint representation this method indicates the presence of a tricritical point[16] at similar β values.

For completeness we present our results for the finite-size behavior of the specific heat C and the modified fourth cumulant V_L at values of $\beta=2.1$ and $\beta=2.25$ (at those values of β we have the most complete data sets on various lattices). In Fig. 6(a) we present our data for the specific heat C, Eq. (5), at $\beta=2.1$. The peak of the specific heat on the 16^4 lattice turns out to be so narrow that its maximum is not hit by the Monte Carlo (MC) runs. However, the curves corresponding to the determination of the specific heat with the Ferrenberg-Swendsen method, Eq. (10), predict a value of the maximum of the specific heat C_{max}. For further illustration we display in Fig. 6(b) the values of C_{max} as obtained by the FS method, including an error estimate, as a function of the lattice size L in logarithmic scales. The divergence of C_{max} with increasing volume is nicely observed for both values of β. At $\beta=2.25$, C_{max} increases for large L according to (6) with α/ν close to 4, consistent with the earlier claim, that at $\beta=2.25$ the Higgs PT is of first order. The linear increase of $\ln C_{max}$ with $\ln L$ at $\beta=2.1$ with a slope different from 4 can be considered as further indication that the phase transition at $\beta=2.1$ is very close to the critical point.

In Figs. 7(a) and 7(b) we compare the cumulant V_L (determined by the FS method) at β values 2.1 and 2.25 on various lattices. While V_L increases with increasing lattice size we expect it to approach the value $\frac{2}{3}$ in the infinite-volume limit for κ away from the phase-transition point, independent of β. However, for a first-order phase transition, we expect that its minimum value V_L^{min} will stay smaller than $\frac{2}{3}$ in the infinite-volume limit on the infinite-volume phase-transition point. Our data at $\beta=2.25$ in Fig. 7(b) indeed show a slower increase of the minimum V_L^{min} with increasing lattice than at $\beta=2.1$ in Fig. 7(a) (see, e.g., the change of V_L^{min} between 12^4 and 16^4 lattices); suggesting a value of V_∞^{min} smaller than $\frac{2}{3}$ at $\beta=2.25$. The effect is however small, and the properties of V_L alone do not allow a determination of the order of the phase transitions at the considered β values.

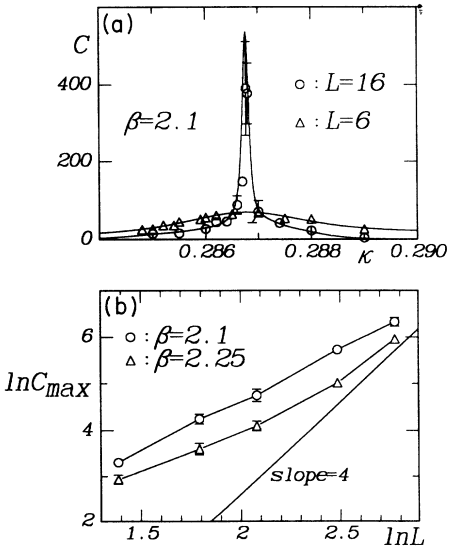

FIG. 6. (a) The specific heat at $\beta=2.1$ on 6^4 and 16^4 lattices. The symbols correspond to single MC runs and determinations of C via Eq. (5). The curves have been obtained with the Ferrenberg-Swendsen method. (b) Dependence of the maximum values C_{max} of the specific heat on the lattice size ($\beta=2.1$ and $\beta=2.25$). The line corresponds to the value $\alpha/\nu=4$.

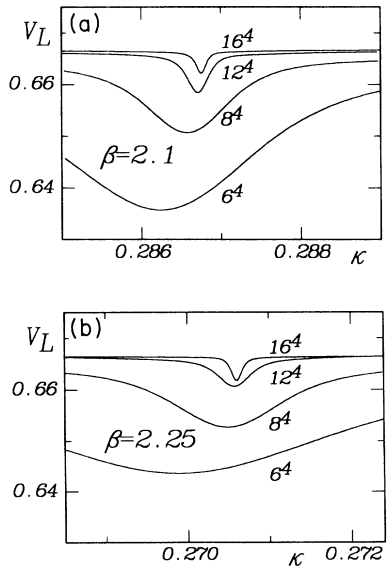

FIG. 7. The fourth-order cumulant V_L at $\beta=2.1$ (a) and at $\beta=2.25$ (b) as obtained with the FS method.

IV. CONCLUSIONS

We performed a careful study of the properties of the Higgs PT at $\lambda=0.5$ at various values of β by varying κ in very fine steps. We conclude from the properties of metastable states observed in the distribution functions $N(E)$, that for $\beta=2.25$ and $\beta=2.4$ the Higgs PT is of first order. Here the gap size is practically unaffected by increasing the lattice size whereas for $\beta=2.1$ a strong decrease of the gap size is observable. We found that the critical end point lies in the interval $1.95<\beta<2.25$ and is probably very close to $\beta=2.1$, i.e., at a value of β substantially larger than suggested by earlier estimates. We studied also the specific heat and the modified fourth-order cumulant V_L at $\beta=2.1$ and $\beta=2.25$ and found a consistent behavior. In the course of our investigation we applied the recently developed multihistogram method of Ferrenberg and Swendsen, which considerably improved our knowledge about the maximum of the specific heat and the minimum of the cumulant in the critical region. We were not able to clarify the existence of a possible tricritical point, but assuming that it exists, it seems most likely to be in the region of β values 2.5–2.7. The Monte Carlo renormalization-group method would seem to offer more hope for resolving this question than do standard Monte Carlo simulations.

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