Coupled Quantum Chains: A bumpy path from 1 to 2 dimensions[1]

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Abstract. A review of the current theoretical knowledge on coupled quantum chains is given, with emphasis on numerical results. These systems occur in the same materials as high T_c superconductors. They allow the study of the dimensional crossover from 1 to 2 dimensions. The behavior of coupled chains, even at isotropic couplings, can be understood in the rung-picture, i.e. the case of large perpendicular couplings. Undoped systems correspond to the Heisenberg spin $\frac{1}{2}$ antiferromagnet, and are well understood. They possess a finite spin gap for an *even* number of coupled chains, and are gapless otherwise. Doped systems are much more difficult to investigate, and only systems with two and three coupled chains have been partially studied.

1 Introduction

Coupled Quantum Chains, also called ladder systems, have been the topic of a lot of recent experimental and theoretical research. This paper presents an overview of the current theoretical understanding of their static behavior, concentrating mainly on numerical studies. It does not attempt to fully cover the literature on the subject. For an excellent review including experimental findings, see [1].

There are several reasons for the theoretical interest in ladder systems. First, real ladder materials do exist, mostly in compounds related to High Temperature Superconductors. Second, ladder systems provide a good way to study the dimensional crossover from 1 to 2 dimensions; this will hopefully also lead to a better understanding of high T_c superconductivity. Third, coupled chains are much easier to investigate, especially numerically, than two-dimensional systems, and one can make use of the extensive knowledge about single chains. Fourth, there is a very interesting analogue of the Haldane conjecture of single chain fame (see below).

The main messages of this paper: The intuitively easy *rung picture* ($J_\perp \gg J$) appears to be valid even at isotropic couplings $J_\perp = J$. For undoped spin-$\frac{1}{2}$ ladders, there is *no spin gap* when the number of chains is *odd*, and a *finite spin gap* when the number of chains is *even*. Doped ladders are not yet well understood.

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In the next chapter we give a brief overview of real ladder-materials. The focus of chapter 3 are *undoped* systems. *Doped* systems correspond to the tJ and the Hubbard model, reviewed in chapter 4. The paper finishes with a summary.

2 Real ladder materials

$(VO)_2P_2O_7$ is a 2-chain material that has been known for some years. Most other real ladder materials are derived from high T_c superconducters. The series $SrCu_nO_{2n-1}$ exhibits $n = 2, 3, ...$ coupled chains, and there are $n = 4, 5, ...$ coupled chains in $(La_2CuO_4)_{n-2}La_2CuO_7$. Fig. 1 shows how these materials are structured. In (a) the two-dimensional precursor is shown, with Cu atoms at solid dots and oxygen atoms at intersections of solid lines. Line defects like in (b) and (c) isolate finite ladders from each other. The coupling between copper atoms across the defect is weak because it involves a 90 degree angle at the oxygen and is frustrated. From this structure it also follows that the ladder materials should have more or less isotropic couplings.

Experimentally these materials are rather difficult to produce. For a review of experimental findings see [2]. Most experiments have been limited to *undoped* materials. Recently, *doped* 2-chain materials have also been produced and analyzed [2].

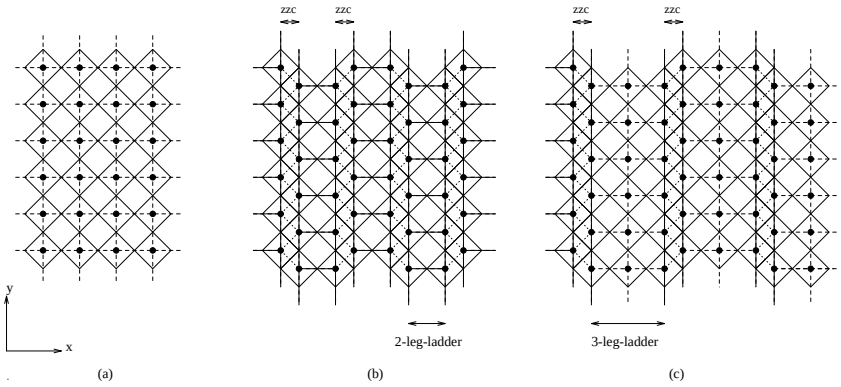

Figure 1: Schematic structure of the CuO planes in ladders derived from high T_c materials. The ladders are separated by linelike defects. (From [11]).

3 Undoped ladders: Heisenberg model

The undoped precursors of high T_c materials are well described by the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg model. Thus this model should also be appropriate for ladder materials. Its Hamiltonian is

$$H = J \sum_{\parallel} \vec{S}_i \vec{S}_j \; + \; J_\perp \sum_{\perp} \vec{S}_i \vec{S}_j \;\; , \tag{1}$$

where J and $J_\perp$ are the couplings parallel and perpendicular to the chain direction, as denoted in fig. 2.

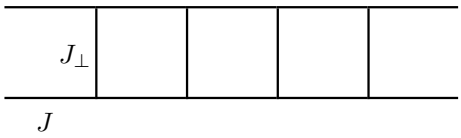

Figure 2: Schematic structure of a 2-chain ladder, with two couplings $J_\perp$ and J.

Both the single chain and the fully two-dimensional model on the square lattice are well understood. The single chain is exactly solvable by Bethe ansatz. It has no energy gap, and no long range order. The two-dimensional model has been extensively studied both by analytical methods (mapping to a nonlinear sigma model) and by numerical studies. There is now little doubt that it also has no energy gap, but long range order. Recent numerical studies of the 2d Heisenberg antiferromagnet [3], using the loop algorithm (see below), were able to make contact with chiral perturbation theory and to determine the free constants in the latter to very high precision.

The *dimensional crossover* from 1 to 2 dimensions in the Heisenberg model is expected to be *far from smooth*: Both the single chain and the two dimensional model are gapless. Yet ladders of *even* width are expected to have an energy gap, whereas those of odd width are again expected to be gapless. This expectation is based on the simple picture of largely independent rungs in the limit $J_\perp \gg J$, which is described in the next section. An open question is whether for very small $J_\perp$ there is qualitatively different behavior, i.e. a phase transition at finite $J_\perp$ from an ungapped to a gapped phase. From existing studies (perturbative as well as numerical) there is no indication for such a phase transition.

The spin-gap behavior is analoguous [4] to that of single spin chains of spin S: when S is half-integer, the latter are known by the Lieb-Shultz-Mattis theorem [5] to be gapless. (A similar proof for odd-width ladders was recently given in [6].) When S is integer, the famous Haldane conjecture [7] predicts a spin gap of order $\exp(-\pi S)$. If the coupling $J_\perp$ between chains in spin ladders

were ferromagnetic, then in the limit $(-J_\perp) \gg J$ the n-chain ladder could be mapped to a single chain of spin $S = \frac{1}{2}n$. However, since $J_\perp$ is actually antiferromagnetic, the relation is not so simple. A possible mapping from antiferromagnetic to ferromagnetic $J_\perp$ is described in section 3.2.

Coupled Heisenberg chains have been extensively studied in the past few years, both analytically and numerically. For an overview of this work, see [1]. In this paper, we shall focus on some selected recent studies.

3.1 Rung picture

The qualitative behavior of spin ladders can be well understood by studying the case $J_\perp \gg J$, as advocated by Dagotto, Rice, and collaborators. The resulting picture remains useful in the isotropic case, and even for $J_\perp < J$.

Let us explain the case of 2 coupled chains. When $J_\perp \gg J$, the *rungs* of the ladder become approximately independent. The groundstate then consists of spin singletts on the rungs ($k_y = 0$, where y is the direction of the rung). The first excited state is that with a single triplett ($k_y = \pi$) on one of the rungs, commonly called a "magnon". It behaves like a hard core boson. The dispersion relation is

$$\epsilon_k = J_\perp + J \cos k_x \quad , \tag{2}$$

and there is a finite gap

$$\Delta \approx J_\perp - J \quad . \tag{3}$$

The higher excited states then form a continuum of states. For $J_\perp \gg J$ the dispersion (2) of the three one-magnon branches does not overlap that of the two-magnon continuum. For $J_\perp = J$ the one-magnon branches are still largely separate from the continuum.

For three coupled chains the rung ground state at large $J_\perp/J$ will be a doublet, like in a single chain. These doublets are weakly coupled by J, leading to a gapless phase, similarly as for single chains. For a recent proof of gaplessness see [6].

3.2 Are the two-chain ladder and the $S = 1$ chain in the same phase ?

The fact that the spin gap of n coupled spin-$\frac{1}{2}$ chains behaves in qualitatively the same way as that of a single spin $S = \frac{1}{2}n$ chain would be very natural for *ferro*-magnetically coupled chains at $-J_\perp \gg J$, since then the spins on each rung would combine to an effective single spin of size $S = \frac{1}{2}n$.

Could it be that the antiferromagnetically coupled ladders ($J_\perp > 0$) and the ferromagentically coupled ones ($J_\perp < 0$) are in the same phase ? Then there should be a continuous path in parameter space from $J_\perp > 0$ to $J_\perp < 0$ which does not go through a phase transition.

For the case of 2 coupled chains, White has given strong arguments in a recent paper [8] that this is indeed the case. Note that the obvious path of just varying $J_\perp$ *does* pass through a transition, since at $J_\perp = 0$ the chains are decoupled and thus gapless, i.e. in a different universality class than the two coupled chains.

White takes the case $J_\perp = J$ and introduces an additional coupling of strength J_2 between sites $(i, 1)$ and $(i + 1, 2)$ $(i = 1, 2, 3, ...)$. When $J_2 = 0$, the original antiferromagnetically coupled chains are recovered. When $J_2 \ll 0$, then sites $(i, 1)$ and $(i + 1, 2)$ are strongly coupled ferromagnetically, so that the spins there will effectively form a single spin of size $S = \frac{1}{2} \cdot 2$.

In a numerical investigation of this model with the density matrix renormalization group, White varied the strength of J_2 while measuring the spin gap. The gap varied only slightly, giving no indication of a phase transition. Thus the gapped phases of two ferromagnetically and two antiferromagnetically coupled chains may indeed be the same. Note however that compared to two antiferromagnetically coupled chains, the modified chain with $-J_2 \gg J$ possesses two additional spins at the ends of the chain, which are coupled only with $J_\perp$ and J.

3.3 Numerical results: DMRG

In DMRG, the lowest states (i.e. zero temperature) of a finite system are iteratively computed while increasing the system size. Isotropically coupled chains were studied by White, Noack, and Scalapino [9], for up to $n = 4$ coupled chains and chain lengths of 24 at $n = 4$. By finite size analysis of the results they do indeed find a spin gap consistent with zero for $n = 1$ and $n = 3$. For the even width ladders they obtain gaps

$$\Delta = 0.504\, J \text{ for 2 chains}, \quad \Delta = 0.190\, J \text{ for 4 chains}, \tag{4}$$

directly from the energy differences of the lowest lying states. (The ladder length at $n = 4$ is rather small, so that the extrapolation to $L = \infty$ might be biased.)

3.4 Numerical results: Quantum Monte Carlo

Detailed results for the thermodynamics of large ladder systems up to 6 chains wide and 100 sites long have been obtained at low temperatures by Frischmuth, Ammon, and Troyer [10, 11]. We shall describe this work in some detail here. It has recently been confirmed and extended [12]. Other Quantum Monte Carlo studies have been performed by Hatano et al. [13] and by Sandvik et al. [14], the latter including dynamics.

The Quantum Monte Carlo studies [10, 12] were made possible by means of the "loop algorithm" [15], which shall be described briefly now. It is a quan-

tum Monte Carlo method in the worldline formulation that employs *nonlocal* changes of worldline configurations and thereby overcomes autocorrelations. Because the Hamiltonian preserves total S_z and couples only locally, all worldlines and all possible *changes* of worldlines (present $\leftrightarrow$ not present, i.e. spin up $\leftrightarrow$ spin down) are located on closed loops, as exemplified in fig. 3. These loops are constructed by stochastic *local* choices of direction, so that no acceptance steps with global weights occur. As a consequence the method is *ergodic*, for example in total magnetization and in worldline winding number.

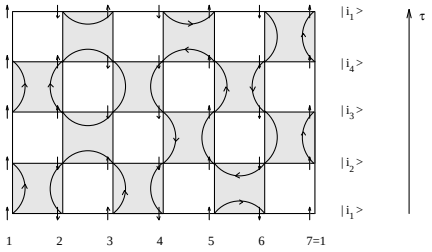

Figure 3: Example of a loop-configuration in the Heisenberg model. The loops connect sites where spins will be flipped together. ([11]).

A further advantage is the existence of "Improved Estimators", which are measurements that average implicitely over 2^{n_l} worldline configurations, where n_l is the number of loops constructed during an update. Thus these improved estimators can have drastically reduced statistical errors. For example, the uniform spin susceptibility can be written

$$\chi \sim \underbrace{\langle \sum_{ij} S_i^z S_j^z \rangle}_{noisy} = \langle \Big(\textstyle\sum_{\text{loops } \alpha} \sum_{\text{sites in } \alpha} S^z \Big)^2 \rangle = \tfrac{1}{4} \underbrace{\langle \sum_{\text{loops } \alpha} W_\alpha^2 \rangle}_{stable} , \quad (5)$$

where W_α is the 'winding number' of loop number α in temporal direction.

The elimination of autocorrelations and the use of improved estimators together resulted in a gain of computer time in [10] of more than 5 orders of magnitude, compared to the standard local worldline QMC method. (For a different good Monte Carlo method, see [14]). This made it possible to go to very large systems of up to 6 chains of length 100, at very low temperatures, down to $T/J = 0.02$ ($\beta J = 50$).

Fig. 4 compares the QMC results for the susceptibility of a single chain with the exact Bethe ansatz result showing that the Monte Carlo calculations are indeed reliable.

The ground state energies of isotropic ladders are given in table 1. They are obtained by extrapolating the Monte Carlo results to zero temperature. One

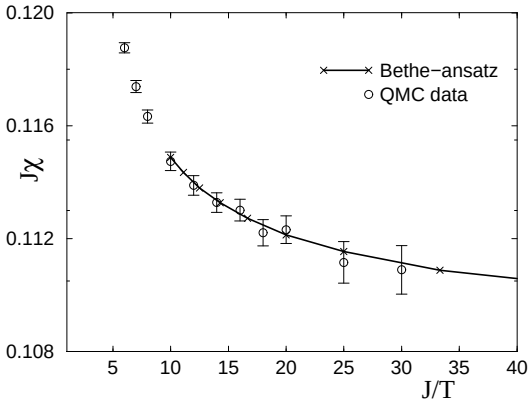

Figure 4: Comparison of QMC with exact results from Bethe ansatz, for a single isotropic chain. (Adapted from [11].)

sees clearly that the ground state energy is a *smooth* function of the ladder width, in contrast to the spin gap.

number of legs	E	reference value
1	-0.4432(1)	-0.44315... (Bethe)
2	-0.5780(2)	-0.578
3	-0.6006(3)	-
4	-0.6187(3)	-
5	-0.6278(4)	-
6	-0.635(1)	-
2D lattice		-0.6693(1)

Table 1: Ground state energies per site for ladders of different widths. ([10].)

The central result is obtained from the spin susceptibility, shown in fig. 5. Both subfigures depict the same data, plotted in (a) against temperature, and in (b) against inverse temperature. The behavior at large temperatures is well described by series expansion. One clearly sees the qualitative difference between ladders of even and of odd width. When there is a gap, then in the limit $T \to 0$ the susceptibility should behave like

$$\chi \stackrel{T\to 0}{\longrightarrow} \frac{e^{-\Delta/T}}{T^{\frac{1}{2}}}. \tag{6}$$

(Here a plausible form [16] for the dispersion relation has been assumed to obtain the power law exponent). On the log-log-plot of fig. 5(b), eq. (6) results in straight lines at large $J\beta$, which are clearly visible in the figure. A fit to

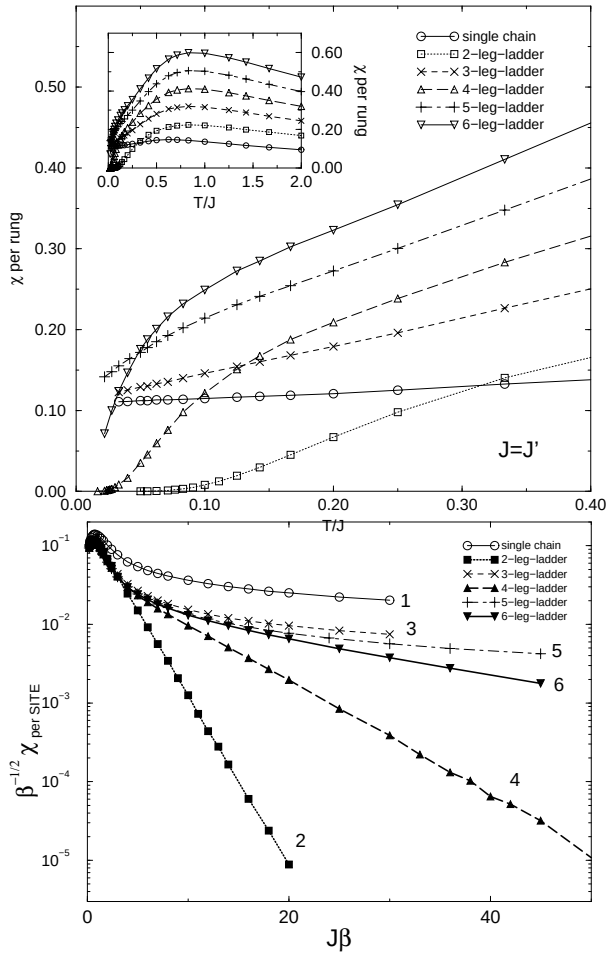

Figure 5: Uniform susceptibility for ladders of different width. Top: χ against temperature ([10]). Bottom: The same data, with $T^{\frac{1}{2}}\chi$ plotted logarithmically against inverse temperature.

their slope gives the gap

$$\Delta = \begin{cases} 0.51(1) & ,2 \text{ legs} \\ 0.17(1) & ,4 \text{ legs} \\ 0.05(1) & ,6 \text{ legs} \end{cases} \tag{7}$$

(with very conservative error bars).

Without a gap, one expects

$$\chi \overset{T\to 0}{\longrightarrow} \text{const} \tag{8}$$

In fig. 5(b), the ladders of odd width still show a slight slope of approximately $0.005J$, which is of the expected magnitude of a finite size effect for ladders of length 100. We can thus conclude that the ladders of odd width are indeed

gapless. Note that the asymptotic $T \to 0$ behavior is reached, as expected, at progressively smaller temperatures of order Δ as the gap decreases.

Further analysis of the ladder systems [11] shows that those of width 2 can be described by a single-magnon picture (as in the rung-picture), even at $J_\perp = J$. Ladders of width 4 can be mapped to two 2-leg-ladders, with one bonding and one anti-bonding branch. The gapless 3 and 5-leg ladders can be mapped approximately to a single chain at an effective coupling J_{eff} at low temperatures. The susceptibility appears to depend *smoothly* on $J_\perp/J$ for $J_\perp > J$. The question whether there is a phase transition at small $J_\perp$ was not addressed in [10, 11].

Comparison of the susceptibility to experimental data gives slightly varying results for J. For two-leg ladders, the experimentally measured value $\Delta \approx 420K$ gives $J \approx 840K$ from eq. (7). For three-leg ladders a fit to data from $Sr_2Cu_3O_5$ results in $J \approx 1100K$. The differences in J may be caused by the effects of residual interactions between different ladders.

In summary, the expected dependence of the gap on ladder width is indeed clearly seen, with gaps in ladders of even width, and gapless behavior in ladders of odd width. Note also that the gaps in eq. (7) are consistent with an exponential decrease with number n of chains, which would be similar to the Haldane behavior $\Delta \sim e^{-\pi S}$ for single chains of spin S. Eq. (7) does not look like the $1/n$ behavior which had been suggested before [9].

4 Doped ladders: tJ model

Doped ladders, i.e. the ladder versions of high temperature superconductors, are especially interesting. Unfortunately, they are very difficult to study both experimentally and theoretically. On the numerical side the results come mostly from exact diagonalization studies of rather small systems of size e.g. 10×2 [19, 20, 21, 22] The relevant models for doped ladders are the tJ and the Hubbard model.

Let us describe the basic issues for the case of the tJ model. Its Hamiltonian is

$$\begin{array}{rll} H = & -\sum_{<ij>,\sigma} t_{ij}(c^+_{i\sigma}c_{j\sigma} + h.c.) & (t_{ij} = t, t_\perp) \\ & +\sum_{<ij>} J_{ij}(\vec{S}_i\vec{S}_j - \frac{1}{4}n_in_j) & (J_{ij} = J, J_\perp) \ . \end{array} \tag{9}$$

At each site there is either an electron of spin up or of spin down, or a hole. At half filling (nondoped case), the tJ model reduces to the Heisenberg model.

A *single* chain in the tJ model is known to be a so called Tomonaga-Luttinger liquid (see below), which has no spin gap, no charge gap, and spin-charge separation.

Experimentally, 2-chain ladders appear to have a spin gap even when doped. The most interesting theoretical issues are to understand this behavior and to

see whether there is a tendency towards superconductivity upon doping.

4.1 Luttinger Liquids

Single chains in the tJ and in the Hubbard model are "Tomonaga-Luttinger Liquids" [17], exhibiting spin-charge separation. Two universality classes are important here. (Actually they are groups of universality classes, since they depend on a parameter K_ρ). In the *Tomonaga-Luttinger liquid (TL)*, both spin and charge are gapless. In the *Luther-Emery liquid (LE)*, only the charge mode is gapless. In both cases, correlations decay with a component that behaves like a power law in inverse distance $1/r$, with an exponent that depends on the single parameter K_ρ. These exponents are given in table 2. Note that at large distances, pairing correlations dominate when $K_\rho > 1$. (However, in a 1-dimensional system at finite temperature there cannot be true long range order. The ladder systems are also 1-dimensional because of their finite width.) Numerical data even on relatively small systems can give information about K_ρ by utilizing conformal field theory.

The case of two coupled Luttinger liquids has been investigated by Schulz [18]. The results are given as "2LL" in table 2. Now both spin and charge modes are massive, and again superconducting correlations are expected to dominate when $K_\rho > 1$.

	TL	**LE**	**2LL**
$2k_f$ SDW	$1 + K_\rho$	exponential	exponential
$2k_f$ CDW	$1 + K_\rho$	K_ρ	exponential
Singlett pairing	$1 + \frac{1}{K_\rho}$	$\frac{1}{K_\rho}$	
Triplett pairing	$1 + \frac{1}{K_\rho}$	exponential	
d-wave pairing			$\frac{1}{K_\rho}$
$4k_f$ CDW	$4K_\rho$	$4K_\rho$	K_ρ

Table 2: Correlation exponents. (For "2LL", the transverse momentum is π).

4.2 tJ model: Rung picture

The rung picture, i.e. the case $J_\perp \gg J$, can also be used to understand coupled tJ-chains [21]. Let us consider the rung picture for 2 coupled chains. In the exact-diagonalization studies described later, the rung picture appears to survive to the case of isotropic coupling (with bound pairs becoming spatially more extended).

In case of half filling, we have the Heisenberg spin model discussed before. If to this we add a single hole, it will behave like a quasiparticle with both spin

and charge. When we add 2 holes, they can either locate on the same rung, or on different rungs. In the ground state, they will be on a single rung to minimize the number of broken rungs of energy cost $J_\perp - t - t_\perp$. Their binding energy corresponds to breaking another rung, i.e.

$$E_{binding} \approx J_\perp - t - t_\perp \ . \tag{10}$$

For energies below $E_{binding}$ we should then expect hole pairs, behaving like hard core bosons with weak attractive interactions $\sim J$, and thus dominant pair correlations, which are "d-wave-like" because of momentum π in rung-direction.

Spin excitations are then gapped. They can either consist of broken pairs of holes with energy $E_{binding}$, or spin-magnons like in the Heisenberg model, with energy $\sim J_\perp$. Since $E_{binding} < J_\perp$, a jump in the gap can be expected between half filling and finite doping. Charge excitations correspond to coherent motion of bound hole pairs and should be gapless. Gapped spin and gapless charge modes are reminiscent of the Luther-Emery liquid.

4.3 tJ model: some results from exact diagonalization

A tentative phase diagram of two coupled tJ chains has been given by Poilblanc et al. [20], for the isotropic case $J_\perp = J$, $t_\perp = t$. At $J/t < 2$ and fillings between about $\frac{1}{2}$ and 1, they do indeed see a phase that resembles the prediction from the rung picture: there is a spin gap, no charge gap, and d-wave like hole pairing. At large $J/t > 4$ they see phase separation between holes and electrons for all fillings. Phase separation minimizes the antiferromagnetic energy. For $J/t < 2$ and filling below $\frac{1}{2}$, the system appears to behave like a Tomonaga-Luttinger liquid, with no gap. At very small filling and $2 < J/t < 4$ there is another phase, with a spin gap and electron pairs.

Interestingly, two coupled chains may behave very differently, depending on whether each *single* chain is integrable or non-integrable. Mila and Poilblanc [23] claim that at the integrable points $J/t = 0$ and $J/t = 2$, and only there, hopping between the chains is "coherent", i.e. it has a periodic component in its time-dependence. This is an area of intense current research.

4.4 Hubbard model

The Hubbard model is a somewhat more realistic description of the doped systems, since it allows two electrons of opposite spin to occupy the same site. Its Hamiltonian is

$$H = -\sum_{<ij>,\sigma} t_{ij}(c^+_{i,\sigma}c_j + h.c.) + U\sum_i n_{i\uparrow}n_{i\downarrow} \ . \tag{11}$$

For this model there exist both weak coupling renormalization group studies and numerical results [24, 25, 26, 27]. At half filling, the dynamical properties have also been investigated [28], with results consistent with a rung picture.

The main features of the phase diagram can most easily be understood from the $U = 0$ case [25], i.e. the free case. Then there are a bonding and an antibonding band, with dispersion

$$\epsilon_k = -(2t \cos k + t_\perp \cos k_\perp), \quad k_\perp = 0, \pi \quad . \tag{12}$$

For $t_\perp < 2t$ the two bands overlap. When some interaction is switched on, there should be Umklapp scattering, producing a Luther-Emery like phase with a spin gap. For $t_\perp > 2t$ the bands are completely separate, and one expects a Tomonaga-Luttinger liquid, without any gap. Thus the simple phase diagram of fig. 6 is obtained.

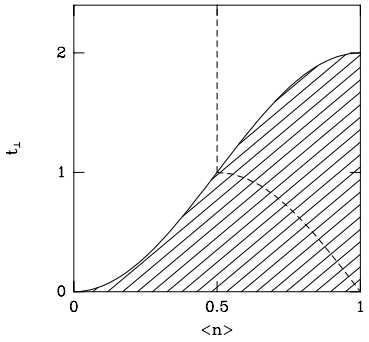

Figure 6: Phase structure for two coupled Hubbard chains at $U = 0$. In the shaded area, both the binding and the antibinding band are occupied. On the dashed line, the lower band is $\frac{1}{2}$ filled. ($t_\perp$ is given in units of t). ([25]).

Balents and Fisher [29] have studied two coupled Hubbard chains by weak coupling renormalization group for infinitesimally small positive U. The resulting phase diagram is shown in fig. 7. Its main structures are very similar to the $U = 0$ case in fig. 6. The phases are denoted as $CnSm$, meaning that there are n gap*less* charge modes and m gap*less* spin modes. The C1S1 phase (no gaps) is like a Tomonaga-Luttinger liquid. C2S2 is the analogue of a Fermi liquid. In the C1S0 phase, there is a spin gap but no charge gap, and d-wave like pairing. Overall the phase diagram exhibits a very rich structure. It is predicted [29] to become even richer as U becomes larger, and remains to be confirmed by independent means.

Noack et al. [24, 25] investigated two coupled Hubbard ladders at *strong* coupling $U = 8$ by means of DMRG, on systems of size 2×8 up to 2×64. Surprisingly, at the couplings investigated they find behavior consistent

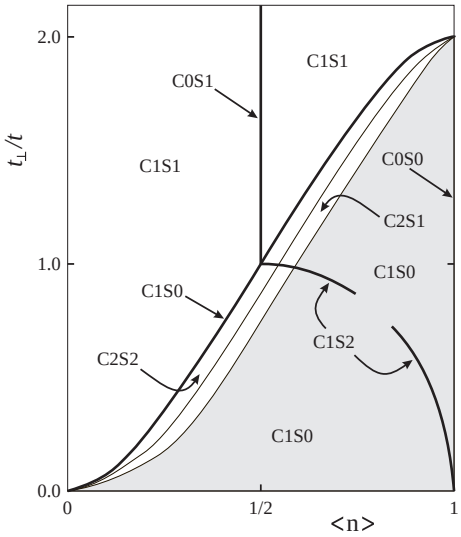

Figure 7: Phase diagram of two coupled Hubbard chains at infinitesimal U. ([29]).

with Balents and Fisher, specifically a C1S0 phase which is Luther-Emery like, showing a spin-gap and d-wave like singlett pairs of holes.

For *three* coupled chains, there is a weak coupling renormalization group calculation by Arrigoni [30], similar to that of Balents and Fisher. The resulting phase diagram, shown in fig. 8, again has a very rich structure of phases. It remains a challenge to verify this structure by other methods.

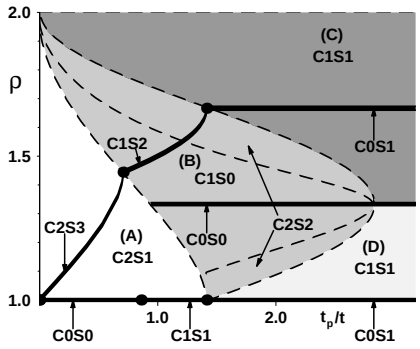

Figure 8: Phase diagram of three coupled Hubbard chains at infinitesimal U. ([30]).

5 Conclusions

Systems of coupled quantum chains, also called ladder systems, are related to the high T_c superconducting materials. They show a very rich and interesting structure, and allow to study the dimensional crossover from one to two dimensions. Much of their behavior can be understood even for the isotropic case in a simple rung picture obtained at large transverse coupling.

Undoped ladders can be described by the Heisenberg spin $\frac{1}{2}$ antiferromagnet, and are rather well understood. For an odd number of coupled chains, they have no spin gap; for an even number of coupled chains, they have a finite gap, which decreases with increasing number of chains. It can be related to the Haldane gap of single spin-S chains. Numerical studies of ladders, including Quantum Monte Carlo studies of up to 6 coupled chains at very low temperature, confirm the expectation and match well to experiment.

Doped ladders may be described by the tJ or the Hubbard model. They are far less well understood, both analytically and numerically. They possess a very rich phase diagram, which includes phases with d-wave like pairing that might evolve into superconducting phases as the number of chains goes to infinity. The phase diagrams proposed from weak coupling renormalization group remain to be explored by other methods.

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