APS Tutorial 7

QSim

Quantum simulation with ultracold atoms

Lecture 1:	Introduction to quantum simulation with ultracold atoms	J. H. Thywissen
Lecture 2:	Hubbard physics with optical lattices	B. DeMarco
Lecture 3:	Ultracold bosons in optical lattices: an overview	A.-M. Rey
Lecture 4:	Quantum simulation & quantum information	I. Deutsch

See pages 22 - 32

Quantum Simulation and Quantum Information

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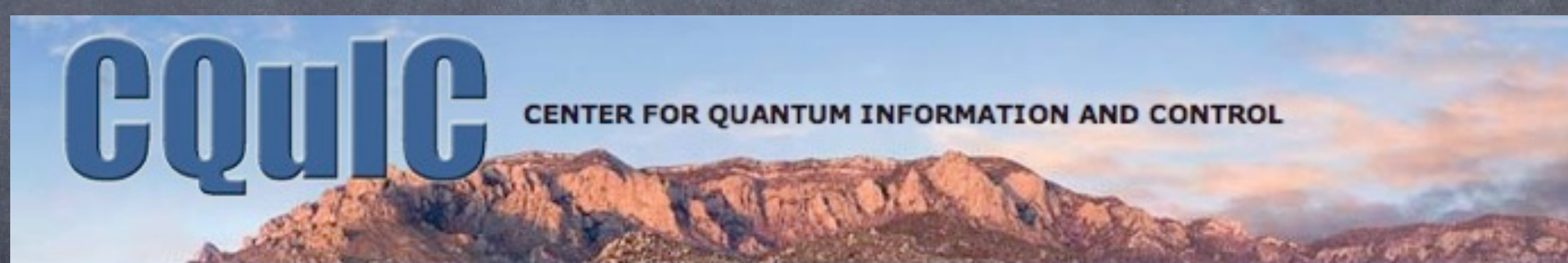

APS physics

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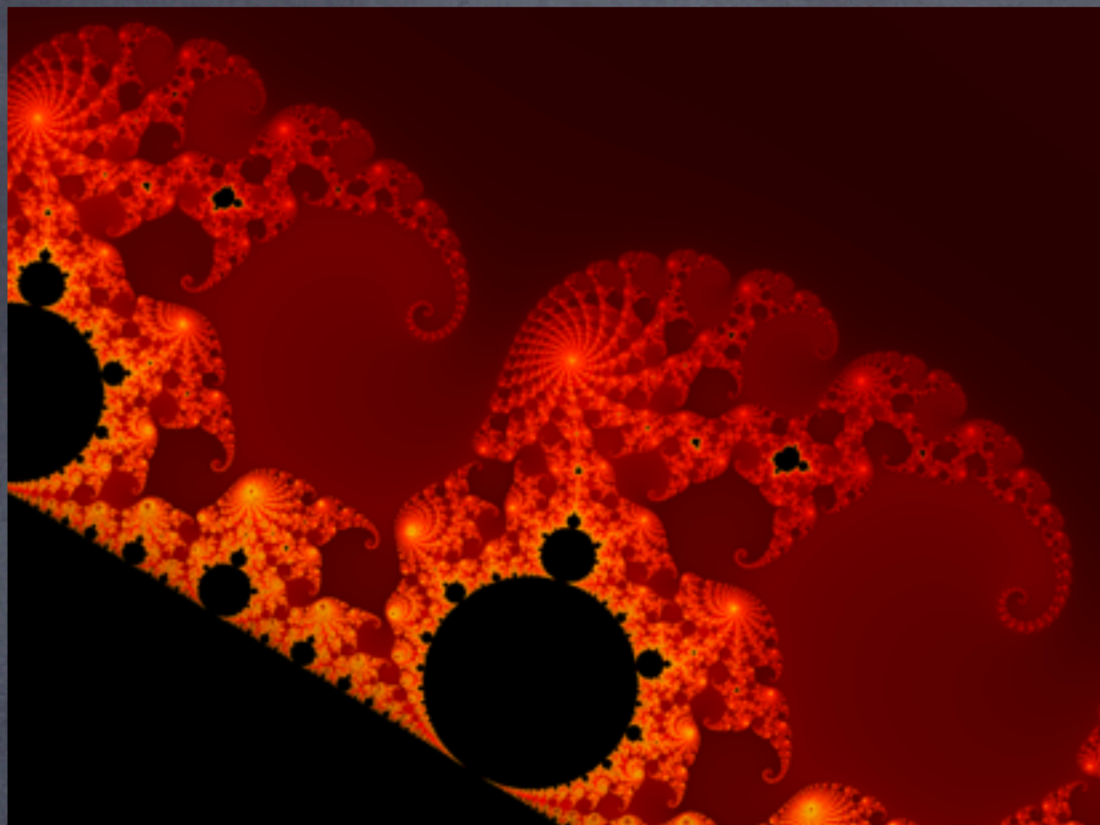
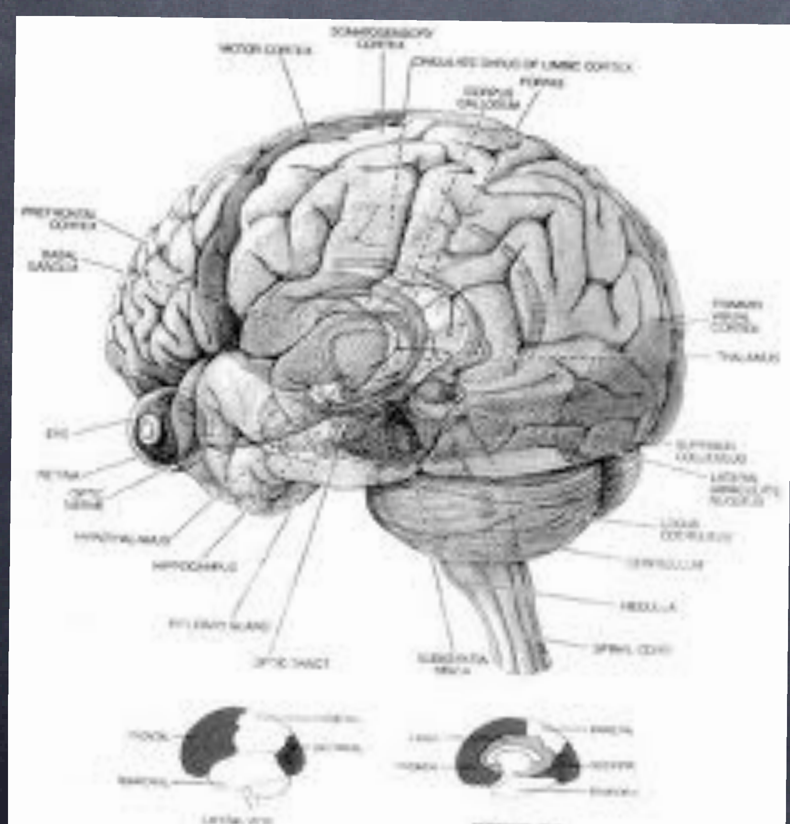
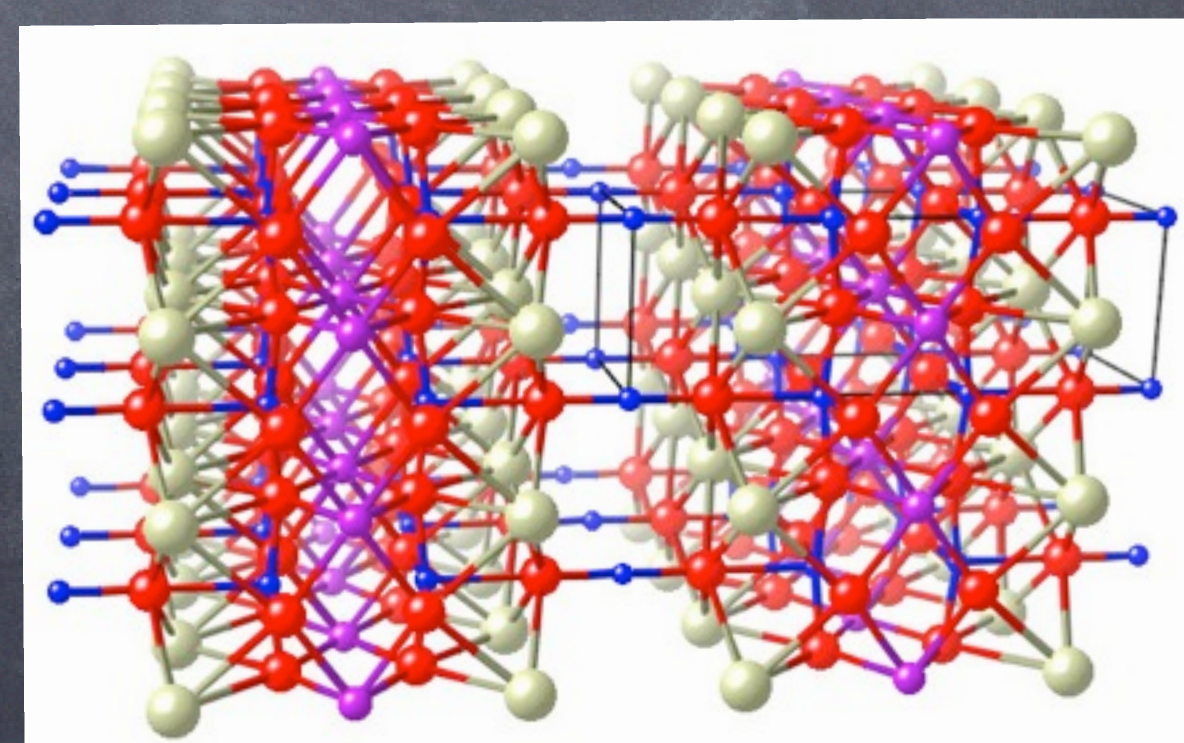
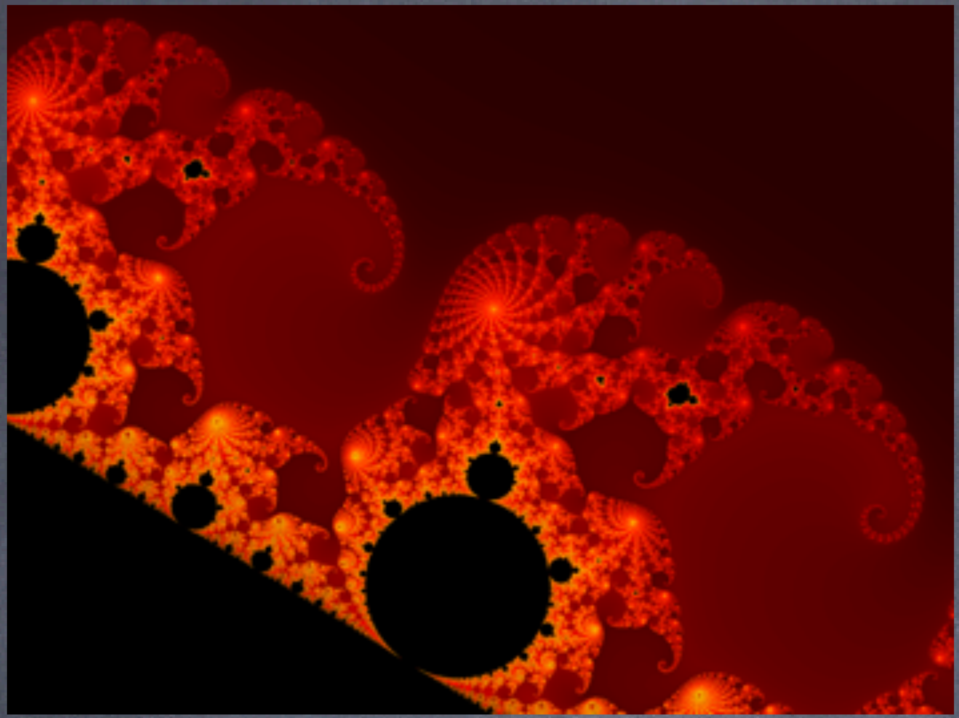
Complexity





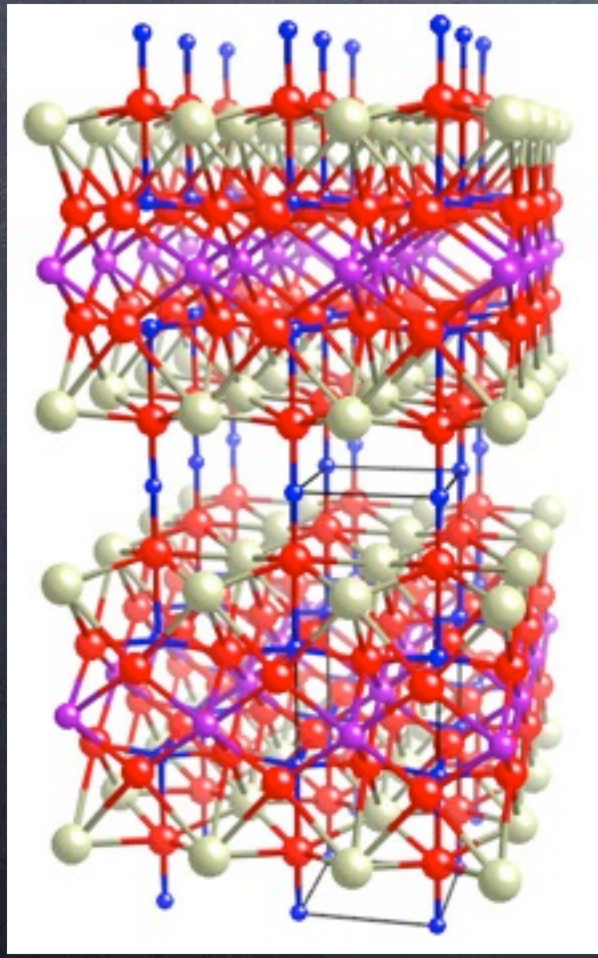


Algorithmic (Kolmogorov) Complexity



Mandelbrot set fractal

- Simply storing the 24-bit color of each pixel in this image would require 1.62 million bits.
- Computer program to generate the image, few lines of code requiring WAY fewer than 1.62 million bits.



High-T_c Superconducting Cuprate

- Complex many-body system of electrons and nuclei.
- Physicists challenge: Find the simplest possible description that captures the essence of the observed phenomenon.
- Mathematical models: analytically solvable or through computer "simulation".
- Find the best mathematical approximation to the physical world.

Models and Simulations

Hubbard model or t-J model

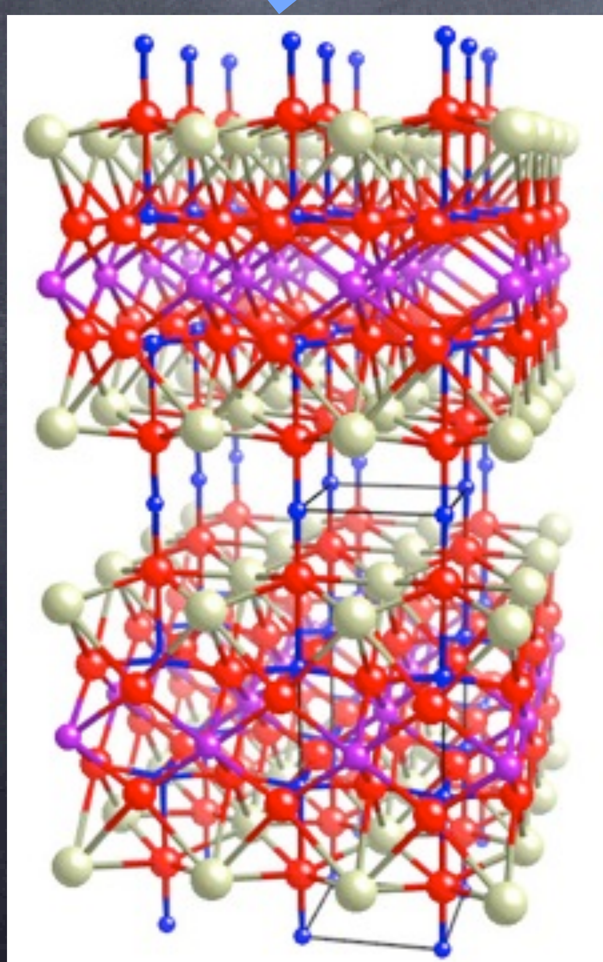
$$H = \sum_{\langle i;j \rangle, s} -t c^{\dagger}_{is} c_{js} + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

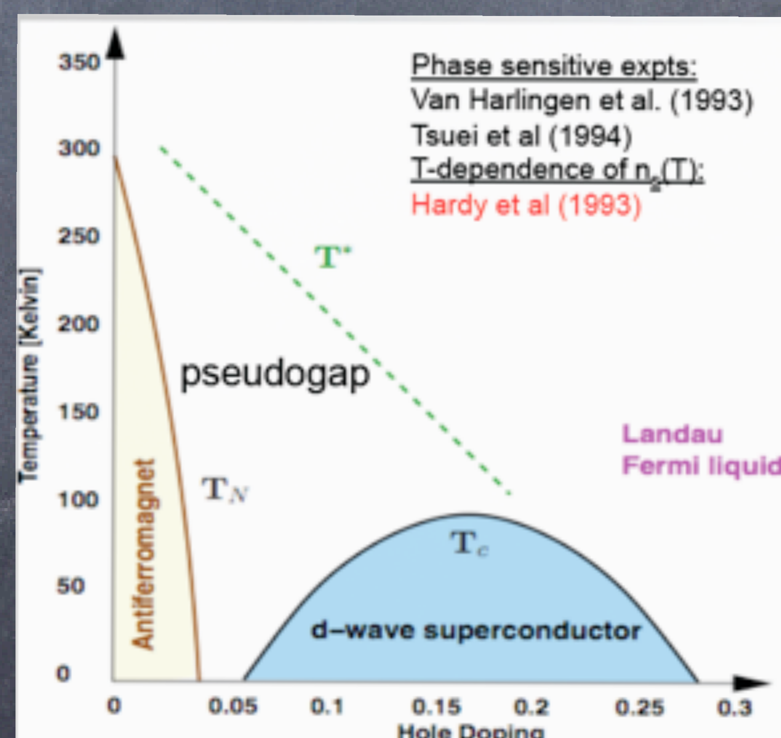
Large on-site interaction U: t-J model

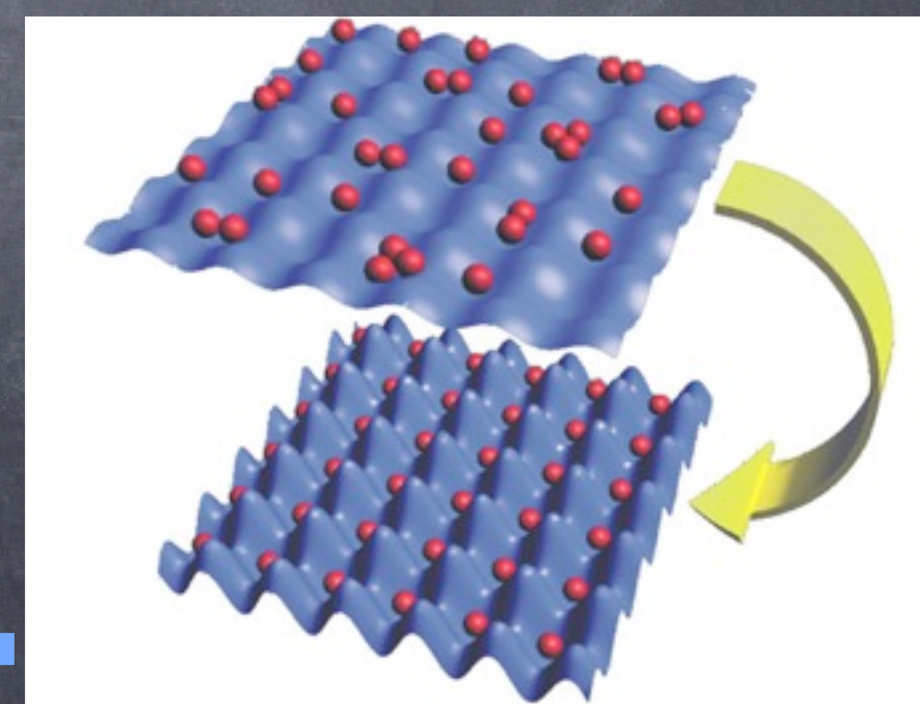
$$H = \sum_{\langle i;j \rangle, s} -t c^{\dagger}_{is} c_{js} + J \sum_{\langle i;j \rangle} \vec{S}_i \cdot \vec{S}_j$$

Model

Simulation by Emulation







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Challenge: Determine the phase diagram associated with a given many-body Hamiltonian.

Hamiltonian

$$H = \sum_{\langle i;j \rangle, s} -t c_{is}^{\dagger} c_{js} + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

State

$$\rho = \frac{e^{-\beta H}}{Z} \underset{\beta \to \infty}{\Rightarrow} \left| \psi_{ground} \right\rangle \left\langle \psi_{ground} \right|$$

Order Parameter:

$$O(\lambda) = Tr\left(\rho(\lambda) \hat{O} \right)$$

Complexity of a many-body quantum state

Hilbert Space description:

$$\mathcal{H} = h^{\otimes N};\ \dim \mathcal{H} = d^N$$

Example state: 4 spin-1/2 particles (16 dim space)

$$\begin{aligned} |\psi\rangle = a^4 |\uparrow\uparrow\uparrow\uparrow\rangle &+ a^3 b\big(|\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\uparrow\uparrow\rangle\big) \\ &+ a^2 b^2 \big(|\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle\big) \\ &+ ab^3 \big(|\downarrow\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\downarrow\downarrow\rangle\big) + b^4 |\downarrow\downarrow\downarrow\downarrow\rangle \end{aligned}$$

$$a = \cos(\theta/2),\ b = \sin(\theta/2)$$

Complex?

$$|\psi\rangle = |\uparrow_\theta\rangle^{\otimes 4}$$

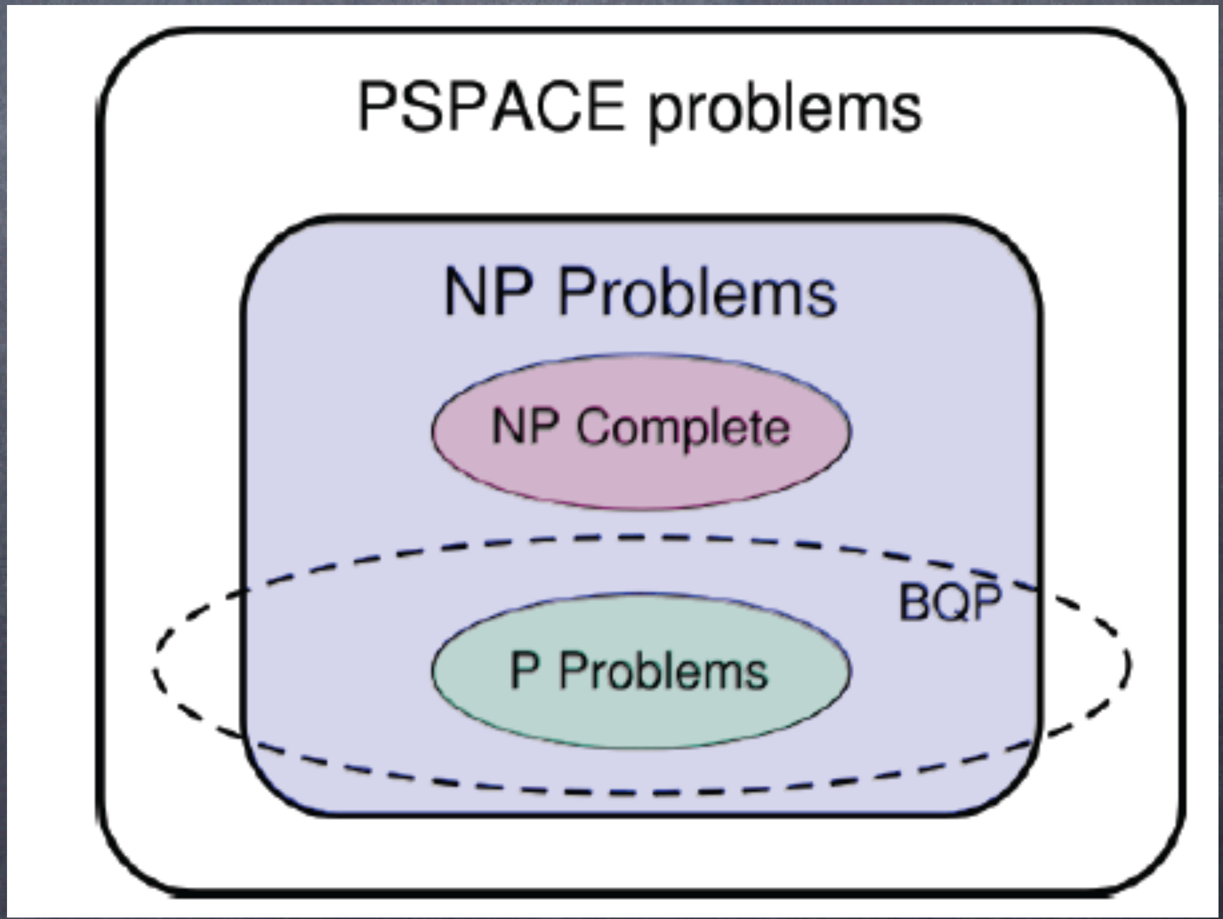
$$|\uparrow_\theta\rangle = a|\uparrow\rangle + b|\downarrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$$

Algorithmically Simple!

Computational Complexity

Quantum Information

Using quantum correlations to solve informationally complex problems



Many-Body Physics and Information

Quantum Information

Using quantum correlations to solve informationally complex problems

Entanglement

Many-Body Physics

Strongly correlated many-body systems

Entanglement

Bipartite Pure-State Entanglement

$$\left|\Psi_{AB}\right\rangle \neq \left|\phi\right\rangle_A \otimes \left|\chi\right\rangle_B$$

$$\rho_A = Tr_B\left(\left|\Psi_{AB}\right\rangle\left\langle\Psi_{AB}\right|\right), \quad \rho_B = Tr_A\left(\left|\Psi_{AB}\right\rangle\left\langle\Psi_{AB}\right|\right)$$

$$Tr\left(\rho_A^2\right) < 1, \quad Tr\left(\rho_B^2\right) < 1$$

Entanglement --> Maximal possible information about the whole (pure state) implies incomplete information about the parts (mixed state).

Entanglement

Quantifying Entanglement: Entropy

$$\rho_A = Tr_B\left(\left|\Psi_{AB}\right\rangle\left\langle\Psi_{AB}\right|\right), \quad \rho_B = Tr_A\left(\left|\Psi_{AB}\right\rangle\left\langle\Psi_{AB}\right|\right)$$

$$E = S(\rho_A) = S(\rho_B) = -\sum_{\mu=1}^{N_S} \lambda_\mu \log \lambda_\mu$$

Singlet: $\left|\Psi_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow_A \downarrow_B\right\rangle - \left|\downarrow_A \uparrow_B\right\rangle\right)$

$$\rho_A = \frac{1}{2}\left|\uparrow_A\right\rangle\left\langle\uparrow_A\right| + \frac{1}{2}\left|\downarrow_A\right\rangle\left\langle\downarrow_A\right| = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \rho_B = \frac{1}{2}\left|\uparrow_B\right\rangle\left\langle\uparrow_B\right| + \frac{1}{2}\left|\downarrow_B\right\rangle\left\langle\downarrow_B\right| = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$E = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \log 2 = 1 \text{ ebit}$$

Schmidt Decomposition

General decomposition into orthonormal basis

$$|\Psi_{AB}\rangle = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} c_{ij}\,|e_i\rangle_A \otimes |f_j\rangle_B$$

Singular value decomposition

$$c_{ij} = \sum_{\mu,\nu=1}^{N_s} U^T_{i\mu}\left(\sqrt{\lambda_\mu}\,\delta_{\mu\nu}\right)V_{\nu j}$$

Schmidt number

$$\Rightarrow |\Psi_{AB}\rangle = \sum_{\mu=1}^{N_s}\sqrt{\lambda_\mu}\underbrace{\left(\sum_{i=1}^{d_A} U_{\mu i}\,|e_i\rangle_A\right)}_{\equiv |u_\mu\rangle_A} \otimes \underbrace{\left(\sum_{j=1}^{d_B} V_{\mu j}\,|f_j\rangle_B\right)}_{\equiv |v_\mu\rangle_B}$$

Schmidt coefficients

Schmidt basis

Schmidt Decomposition

$$|\Psi_{AB}\rangle = \sum_{\mu=1}^{N_s}\sqrt{\lambda_\mu}\,|u_\mu\rangle_A \otimes |v_\mu\rangle_B$$

Schmidt Decomposition

Schmidt Decomposition

$$\left|\Psi_{AB}\right\rangle = \sum_{\mu=1}^{N_s} \sqrt{\lambda_\mu} \left|u_\mu\right\rangle_A \otimes \left|v_\mu\right\rangle_B$$

Marginal density operators

$$\rho_A = \sum_{\mu=1}^{N_s} \lambda_\mu \left|u_\mu\right\rangle\left\langle u_\mu\right| \qquad \rho_B = \sum_{\mu=1}^{N_s} \lambda_\mu \left|v_\mu\right\rangle\left\langle v_\mu\right|$$

Entanglement

$$E = S(\rho_A) = S(\rho_A) = -\sum_{\mu=1}^{N_s} \lambda_\mu \log \lambda_\mu$$

The Schmidt decomposition quantifies the "complexity" of the state

Many-body Complexity and Entanglement

VOLUME 91, NUMBER 14 PHYSICAL REVIEW LETTERS week ending 3 OCTOBER 2003

Efficient Classical Simulation of Slightly Entangled Quantum Computations

Guifré Vidal

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(Received 25 February 2003; published 1 October 2003)

“Clearly, if a quantum device is to offer an exponential speedup with respect to classical computations, then it must involve dynamics that cannot be efficiently simulated classically.”

Many-body Complexity and Entanglement

N, d-level systems

$$|\Psi\rangle = \sum_{i_1 i_2 \cdots i_n} c_{i_1 i_2 \cdots i_n} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle$$

$$d^N - \text{parameters: } c_{i_1 i_2 \cdots i_N}$$

Schmidt decomposition for arbitrary biparite division A+B

$$|\Psi\rangle = \sum_{\mu=1}^{N_{AB}} \lambda_\mu^{[A,B]} |\Phi_\mu^{[A]}\rangle \otimes |\Phi_\mu^{[B]}\rangle$$

Information content

$$E = \log(N_{AB}^{\max}), \qquad O\left(n2^E\right) \text{ parameters needed to specficy } |\Psi\rangle$$

$$c_{i_1 i_2 \cdots i_n} = Tr(A^{(i_1)}[1] A^{(i_2)}[2] \cdots A^{(i_n)}[n])$$

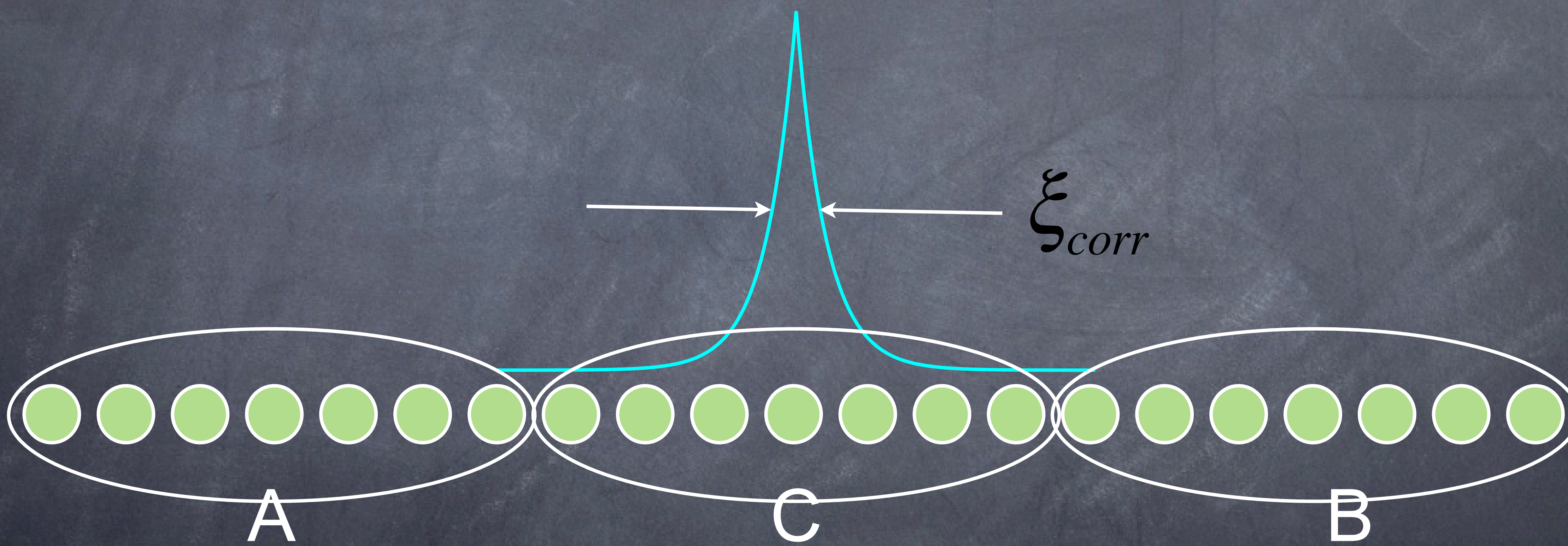
Matrix for the i_k component of k^{th} subsystem: $A_{\mu\nu}^{(i_k)}[k] = U_{\mu\nu}^{(i_k)}[k]\, \lambda_\nu^{[k,n-k]}$

$$E \le O(\log n) \Rightarrow \text{Efficient representation}$$

Entanglement and Local Systems

Spin chain (Ising-like model)

- Local interactions --> Finite correlations away from critical point (point of second order phase transition)



$$\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle \approx \exp\left(-\frac{l_{AB}}{\xi_{corr}} \right)$$

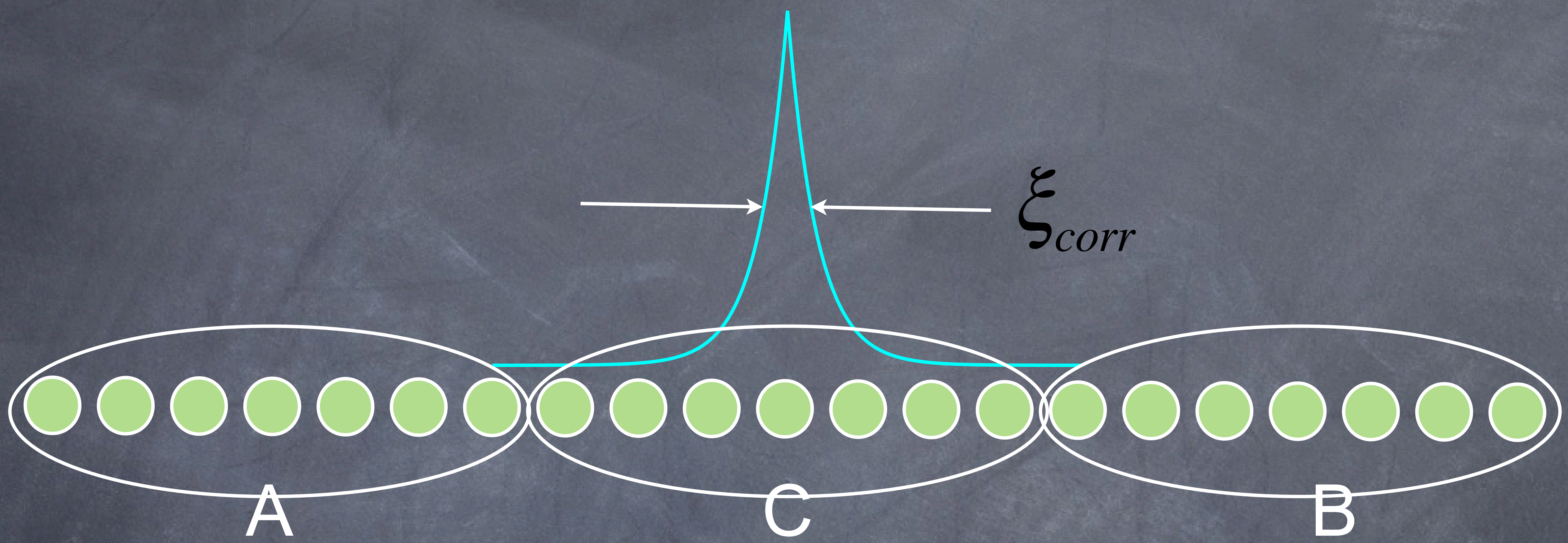
Entanglement and Local Systems

- Local interactions --> Short-range correlations away from critical point (second order phase transition).
- Short-range correlations --> Limited entanglement --> Simple representation with limited information.
- Critical point --> Diverging correlation length

$$\boxed{\xi_{corr} \sim |\lambda - \lambda_{crit}|^{-\nu}}$$

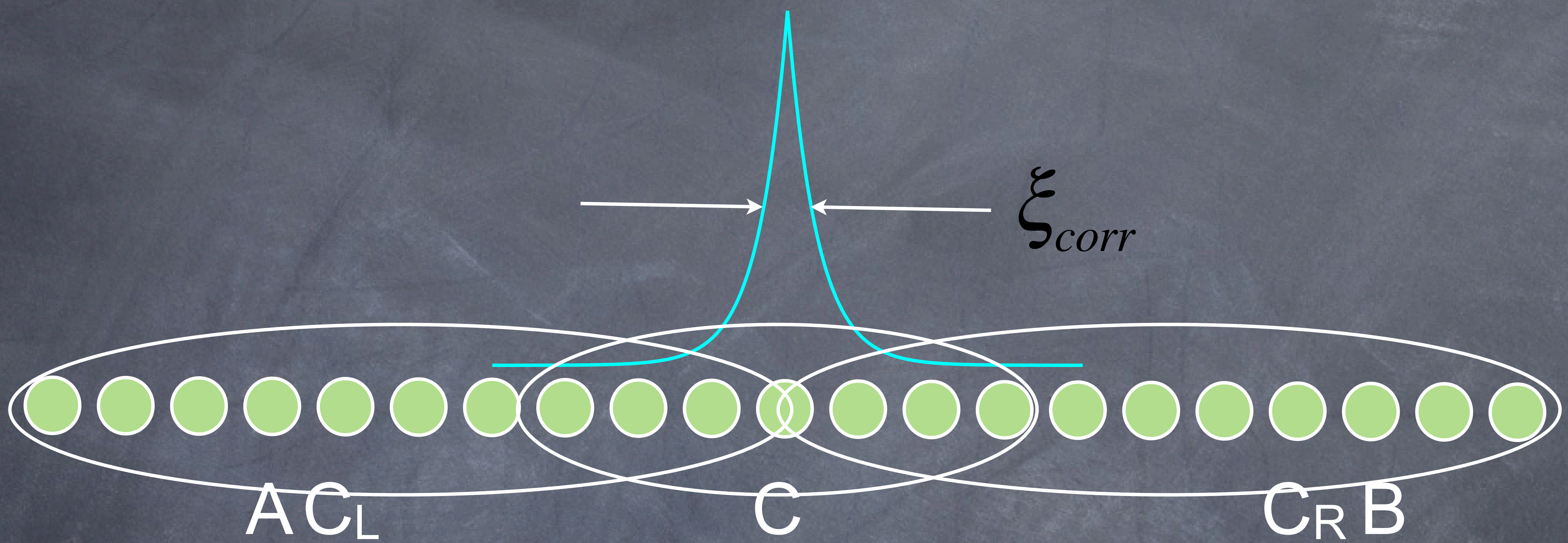
- Critical point --> Larger entanglement --> Complex representation --> difficult to simulate.

Matrix Product States



$$\rho_{AB} = Tr_C\left(\left|\Psi_{ABC}\right\rangle\left\langle\Psi_{ABC}\right|\right) \approx \rho_A \otimes \rho_B$$

Matrix Product States



$$\rho_{AB} = Tr_C\left(\left|\Psi_{ABC}\right\rangle\left\langle\Psi_{ABC}\right|\right) \approx \rho_A \otimes \rho_B$$

$$\left|\Psi_{AC_L}\right\rangle \otimes \left|\Psi_{AC_R}\right\rangle \approx I_A \otimes U_C \otimes I_B \left|\Psi_{ACB}\right\rangle$$

$$\left|\Psi_{ACB}\right\rangle \approx \sum_{\mu,\nu,i} A^i_{\mu\nu} \left|\psi_\mu\right\rangle_A \otimes \left|\phi_i\right\rangle_C \otimes \left|\chi_\nu\right\rangle_B$$

Matrix-Product State!

Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems

F. Verstraete[a*], V. Murg[b] and J.I. Cirac[b]

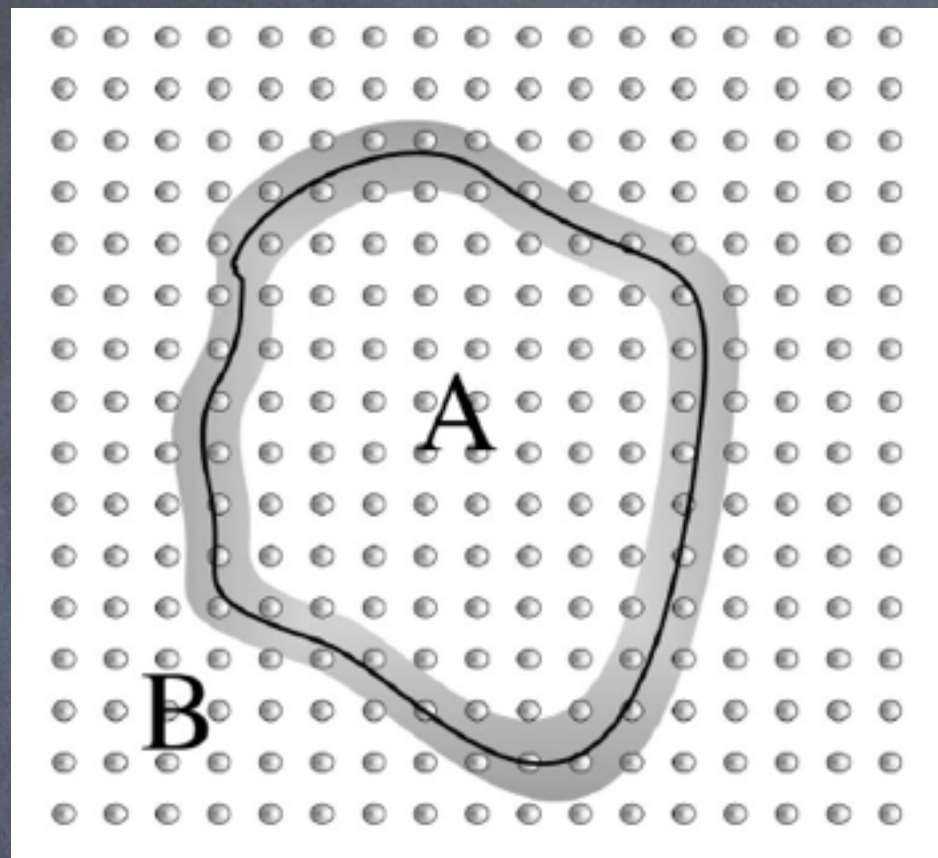
Advances in Physics
Vol. 57, No. 2, March–April 2008, 143–224

Matrix Product State:

$$|\Psi\rangle = \sum_{i_1 i_2 \cdots i_n} Tr\left(A^{(i_1)}[1] A^{(i_2)}[2] \cdots A^{(i_n)}[n]\right) |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle$$

- Variational wave function at the heart of Density-Matrix Renormalization Group (DMRG).
- For 1D gapped (noncritical) systems, form a faithful representation with small dimensional matrices (limited entanglement).
- Generalization to 2D -- Projected Entangled Pairs (PEPS)
- Fermions in 2D? Unsolved whether there exists efficient representation.

Area Laws



$$S(\rho_A) \sim cL^{D-1}$$

- For gapped quantum spin systems, the entanglement scales as the size of boundary.
- Critical systems: 1D S~cLog(L)
- Higher dimensional critical systems? Fermions?

Models and Simulations

Hubbard model or t-J model

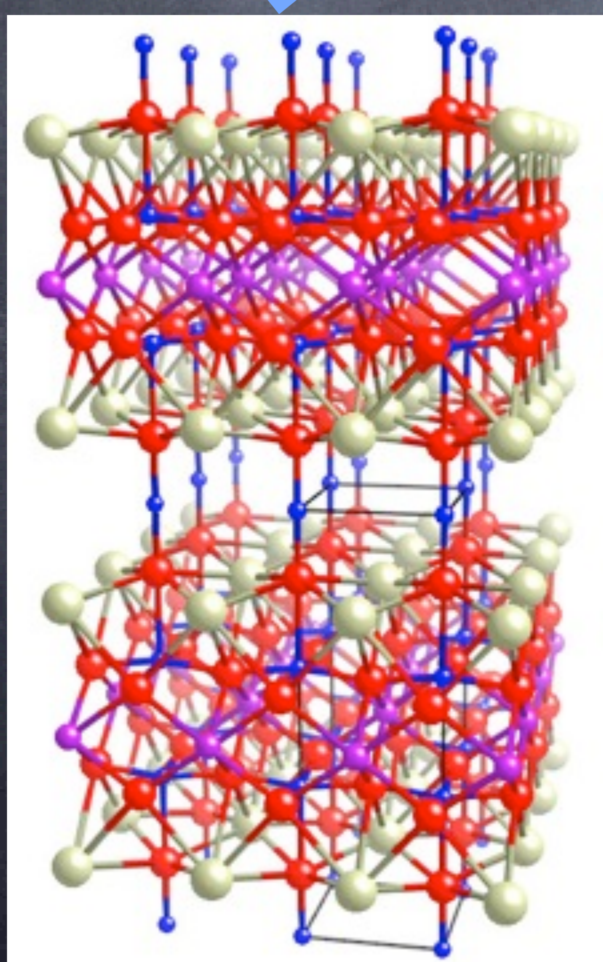
$$H = \sum_{\langle i;j\rangle,s} -tc^{\dagger}_{is}c_{js} + U\sum_{i} n_{i\uparrow}n_{i\downarrow}$$

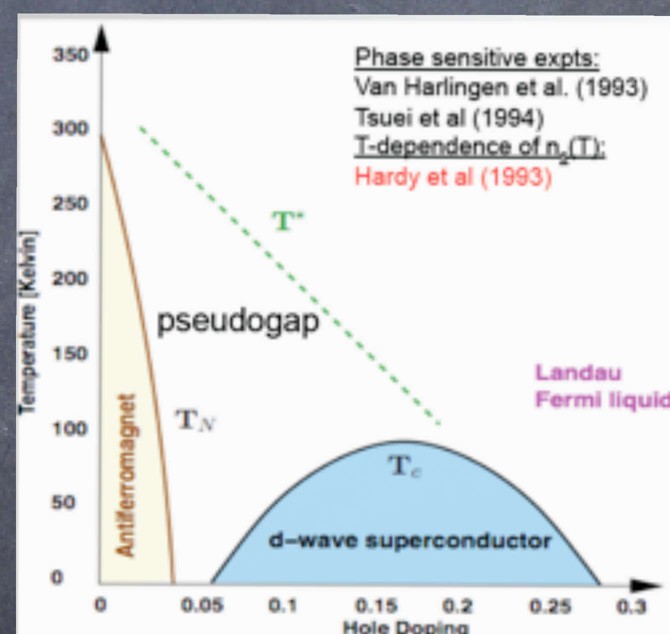
Large on-site interaction U: t-J model

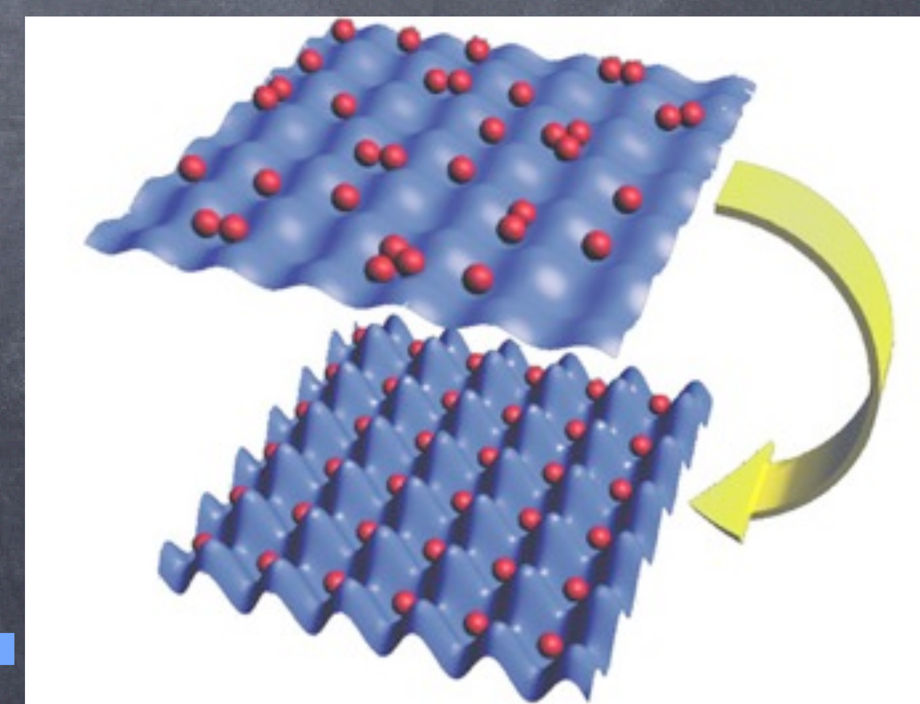
$$H = \sum_{\langle i;j\rangle,s} -tc^{\dagger}_{is}c_{js} + J\sum_{\langle i;j\rangle} \vec{S}_i \cdot \vec{S}_j$$

Model

Simulation by Emulation







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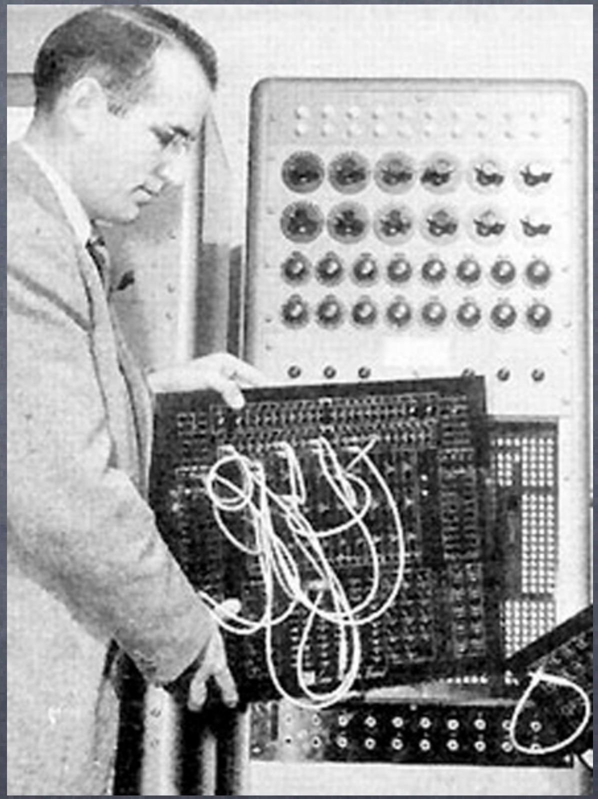
Requirements on a Quantum Simulator

- Quantum Simulator: Physical system should be faithfully described by the desired model.
- Most challenging (not accessible by classical computation) for "complex quantum states".
- Complex quantum states (large information content) have substantial entanglement.
- How robust is a quantum simulator, and how can we test its veracity?

Quantum Simulation: Analog vs. Digital

Analog: Finding the solution through the laws of physics

- Differential equations of motion of masses on springs with dashpots etc. equivalent to electrical voltages in circuits with resistors, capacitors etc.



- Computational complexity of analog computer difficult to assess. As problem size grows the signal disappears into the noise.

Models and Simulations

Hubbard model or t-J model

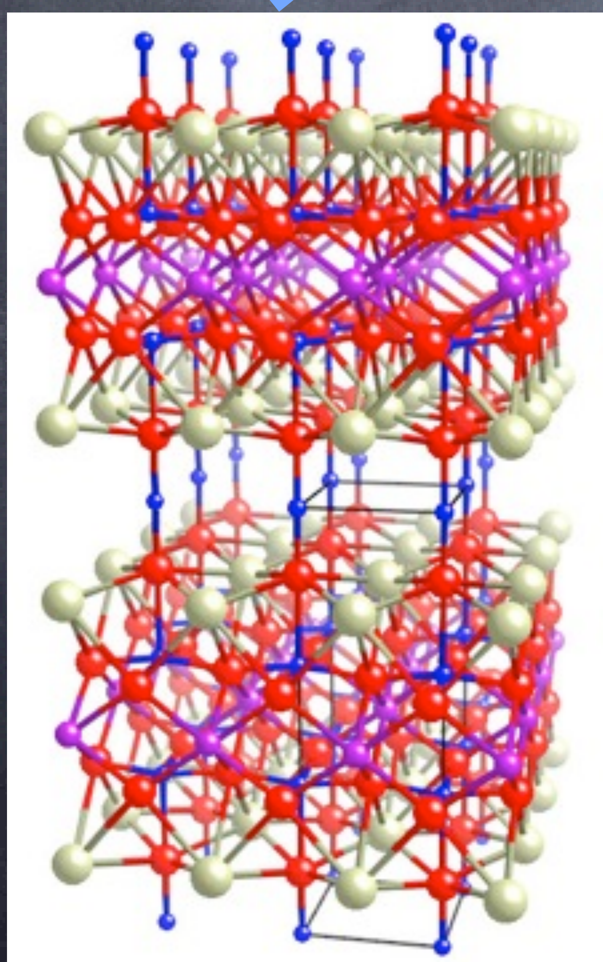
$$H = \sum_{\langle i;j \rangle, s} -t c^{\dagger}_{is} c_{js} + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

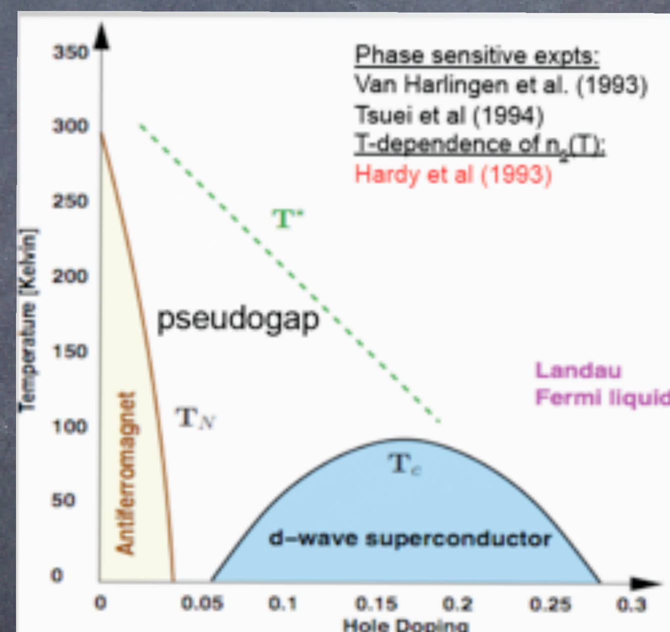
Large on-site interaction U: t-J model

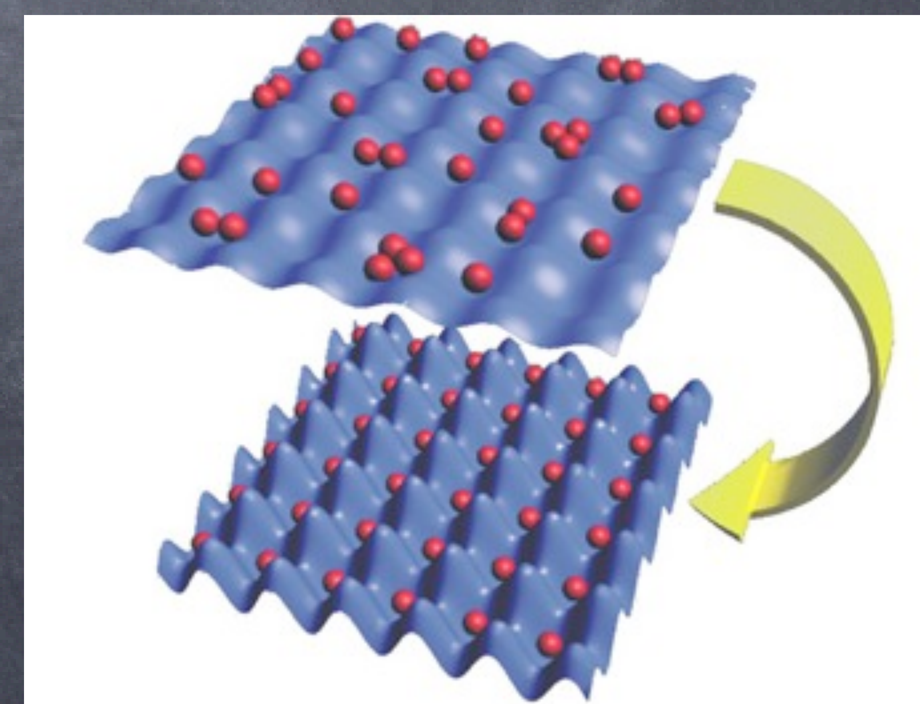
$$H = \sum_{\langle i;j \rangle, s} -t c^{\dagger}_{is} c_{js} + J \sum_{\langle i;j \rangle} \vec{S}_i \cdot \vec{S}_j$$

Model

Simulation by Emulation







Quantum Information Processing: Analog or Digital?

The physical nature of information

Rolf Landauer [1]
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Received 9 May 1996
Communicated by V.M. Agranovich

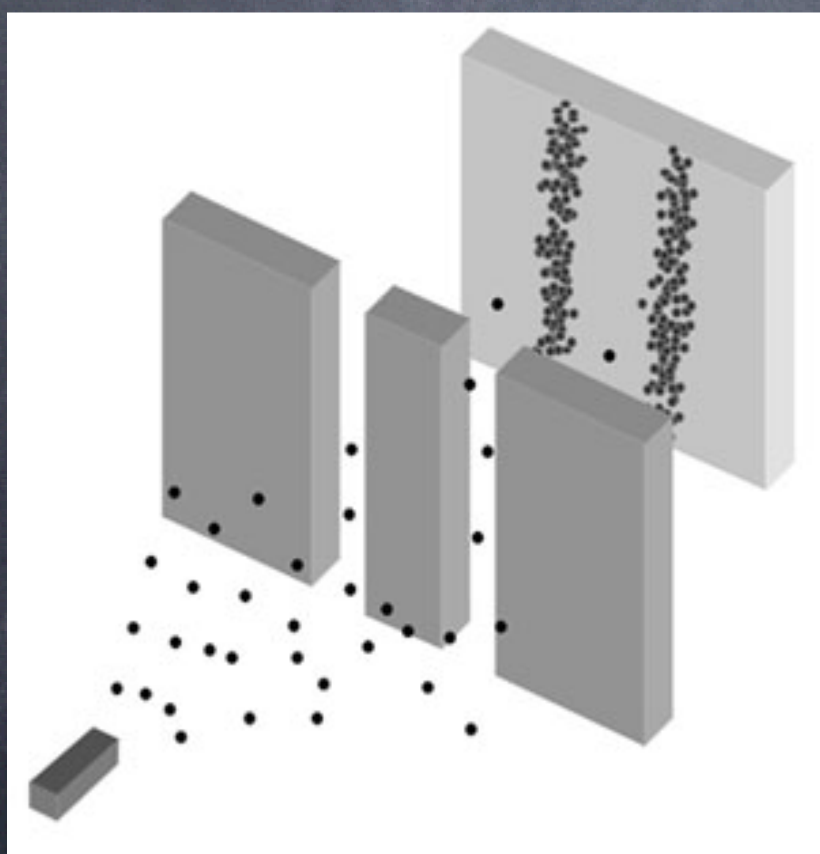
3. Quantum parallelism: A return to analog computation

An analog computer can do much more per step than a digital computer. But an analog computer, in which a physical variable such as a voltage can take on any value within a permitted range, does not allow for easy error correction. Therefore, in the analog computer errors, due to unintentional imperfections in the machinery, build up quickly and the procedure can go through only a few successive steps before the errors accumulate prohibitively. A digital computer, by contrast, allows only a *0* or *1*. That permits us to restore signals toward their intended values, before they drift

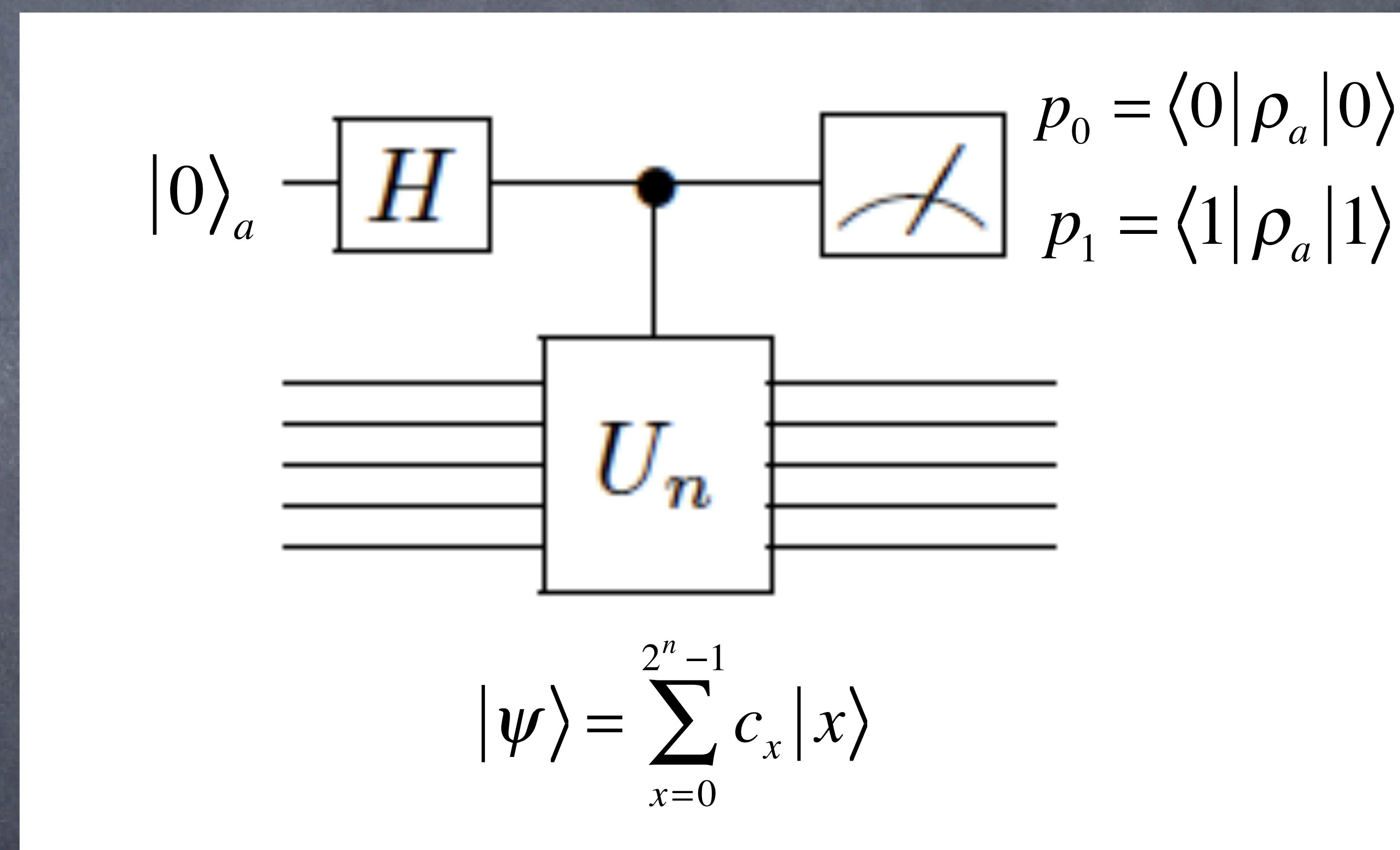
far away from that. In typical digital logic the signal is restored toward the power supply voltage or ground at every successive stage. This is what permits us to go through a tremendous number of successive digital steps, and this has given the digital computer its power. In quantum parallelism we do not just use *0* and *1*, but all their possible coherent superpositions. This continuum range, which gives quantum parallelism its power, also gives it the problems of analog computation, a point first explicitly stated by Peres [16]. If we

Quantum Information: Analog vs. Digital?

Wave-Particle Duality



Analog-Digital Duality



Quantum Error Correction and Fault-Tolerance

- Digital nature of quantum information allows us to discretize the errors

Quantum error correcting code

$$|0\rangle_{\text{code}} = \frac{1}{\sqrt{8}}\left(\sum_{\substack{\text{even } v\\ \in \text{ Hamming}}} |v\rangle\right)$$
$$= \frac{1}{\sqrt{8}}\Big(|0000000\rangle + |0001111\rangle + |0110011\rangle + |0111100\rangle + |1010101\rangle + |1011010\rangle + |1100110\rangle + |1101001\rangle\Big)$$

$$|1\rangle_{\text{code}} = \frac{1}{\sqrt{8}}\left(\sum_{\substack{\text{odd } v\\ \in \text{ Hamming}}} |v\rangle\right)$$
$$= \frac{1}{\sqrt{8}}\Big(|1111111\rangle + |1110000\rangle + |1001100\rangle + |1000011\rangle + |0101010\rangle + |0100101\rangle + |0011001\rangle + |0010110\rangle\Big)$$

- Can detect errors without detecting the "quantum path".
- Process of error-correction is fault-tolerant when the errors are below a given threshold. $p_{\text{Steane}} \sim 10^{-5}$

but requires extremely large numbers of qubits

The Common Lore of Quantum Simulation

108 2 OCTOBER 2009 VOL 326 **SCIENCE** www.sciencemag.org

Quantum Simulators

Iulia Buluta[1] and Franco Nori[1,2]*

in general, quantum simulations do not require either explicit quantum gates or error correction, and less accuracy is needed. Thus, quantum simulation is typically less demanding than quantum computation. Even with tens of qubits (*4–6*), one could already perform useful quantum simulations, whereas thousands of qubits would be required for factorizing even modest numbers using of Shor's algorithm.

Question:

Why does Shor's factoring algorithm require quantum error correction, but a useful quantum simulation (i.e., one not efficiently simulatable on a classical computer) not?

Robustness of information: What do we measure?

- Typical quantum algorithm (e.g. Shor): Measure P_x in computational basis to due answer. Requires robustness of 2^n probabilities.

$$P_x = \left|\langle x|\Psi\rangle\right|^2$$

- Typical quantum simulation: Measure local correlation function to determine the order parameter, e.g., quantum magnetism:

$$C = \sum_{neighbors} \langle \sigma_z^i \sigma_z^j \rangle$$

Question

When is C not efficiently calculable on a classical computer, and when it is not, how sensitive is it to errors in the quantum many-body state?

Is Nature Quantum Complex?

Question

Does nature make use of exponentialamounts of entanglement especially at finite temperature and with finite imperfection?

Fundamental Question

- Under what condition is the quantum state of a many-body system sufficiently robust that we can use it to perform a useful quantum simulation without digital encoding for error correction, and when it is that robust, could we have obtained that information otherwise in an efficient calculation on a classical computer?

- Solution 1: An analog quantum simulator is not reliable and can only capture finite entanglement scales --> Need to encode digitally in order to correct errors.

- Solution 2: An analog quantum simulator can solve classically intractable problems --> We should take advantage of this robustness is all possible ways for quantum computation.

Next Frontier in Complexity Theory!

References: Below are mainly review articles and books. See also the references therein.

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IV. Entanglement and Quantum Phase Transitions

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VI. Editorial opinion

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