

8. Linear Algebra

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8.1 Constructing Matrices

M = list = { {a11,a12,a13,...a1n}, {a21,a22,a23, ..., a2n},, {am1,am2,am3, ..., amn} }	
M = Table[f[i,j], {i,m}, {j,n}]	build a (n x m) matrix, whose element M[i,j] is given by the value f[i,j]
A = Array[a, {m,n} }	build a (n x m) matrix, whose element a[i,j] is the symbol a[i,j]
DiagonalMatrix[list]	generate a diagonal matrix with the elements of list on the main diagonal
IdentityMatrix[n]	generate the (n x n) Identity Matrix
TableForm[list]	Prints list with the elements of list arranged in a regular array
MatrixForm[list]	Prints list as a matrix

```
A = {{1,2,3,4}, {2,3,0,-5}, {2,-1,1,1},{-2,2,0,-5} }  
{ {1, 2, 3, 4}, {2, 3, 0, -5}, {2, -1, 1, 1}, {-2, 2, 0, -5} }
```

MatrixForm[A]

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

```
M = Table[m[i,j], {i,2}, {j,3}]
```

```
{ {m[1, 1], m[1, 2], m[1, 3]}, {m[2, 1], m[2, 2], m[2, 3]} }
```

MatrixForm[M]

$$\begin{pmatrix} m[1, 1] & m[1, 2] & m[1, 3] \\ m[2, 1] & m[2, 2] & m[2, 3] \end{pmatrix}$$

```
f[i_,j_] = i/j;
```

```
MF = Table[f[i,j], {i,3}, {j,3} ]
```

```
{{1, 1/2, 1/3}, {2, 1, 2/3}, {3, 3/2, 1}}
```

TableForm[MF]

```
1  1/2  1/3  
2  1    2/3  
3  3/2  1
```

```
Clear[a, aa]
```

```
aa = Array[a, {3, 4}]
```

```
{ {a[1, 1], a[1, 2], a[1, 3], a[1, 4]},  
  {a[2, 1], a[2, 2], a[2, 3], a[2, 4]}, {a[3, 1], a[3, 2], a[3, 3], a[3, 4]} }
```

MatrixForm[%]

$$\begin{pmatrix} a[1, 1] & a[1, 2] & a[1, 3] & a[1, 4] \\ a[2, 1] & a[2, 2] & a[2, 3] & a[2, 4] \\ a[3, 1] & a[3, 2] & a[3, 3] & a[3, 4] \end{pmatrix}$$

a[i_, j_] := ToExpression["a" <> {ToString[i] , ToString[j]}]

MatrixForm[aa]

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

Clear[a, aa]

aa = Array[a, {3, 4}];

a[i_, j_] := Subscript[a, i, j]

MatrixForm[aa]

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$$

Di = DiagonalMatrix[{2,1,0,-1,-2}];

MatrixForm[Di]

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

MatrixForm[IdentityMatrix[3]]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

MatrixForm[0 IdentityMatrix[3]]

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

MatrixForm[Table[0, {2}, {4}]]

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Array[0, {2, 4}]

{{0[1, 1], 0[1, 2], 0[1, 3], 0[1, 4]}, {0[2, 1], 0[2, 2], 0[2, 3], 0[2, 4]}}

MatrixForm[Table[If[i > j,0,1], {i,3},{j,3}]]

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

MatrixForm[Table[If[i == j,0,1], {i,3},{j,3}]]

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

```
MatrixForm[Table[If[i <= j,0,1], {i,3},{j,3}]]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

8.2 Getting Pieces of Matrices

<code>m[[i,j]]</code>	the <i>i,j</i> -th element
<code>m[[i]]</code>	the <i>i</i> -th row
<code>Transpose[m][[i]]</code>	the <i>i</i> -th column
<code>m[{{i1,...,ir},{j1,...,js}}]</code>	the <i>r x s</i> submatrix of m with elements having row indices ik and column indices jk
<code>m[[Range[i0,i1], Range[j0,j1]]]</code>	a submatrix with elements having row and column indices respectively, in the ranges given
<code>Diagonal[m]</code>	gives a list containing the elements of the leading diagonal of the matrix m
<code>Diagonal[m, k]</code>	gives a list containing the elements of the <i>k</i> -th diagonal of the matrix m

`Diagonal` works also on rectangular matrices.

```
A = {{1,2,3,4}, {2,3,0,-5}, {2,-1,1,1},{-2,2,0,-5}}; MatrixForm[A]
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

```
Diagonal[A, -3]
```

```
{-2}
```

```
A[[3,2]]
```

```
-1
```

```
A[[2]]
```

```
{2, 3, 0, -5}
```

```
B = Transpose[A]; MatrixForm[B]
```

$$\begin{pmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & -1 & 2 \\ 3 & 0 & 1 & 0 \\ 4 & -5 & 1 & -5 \end{pmatrix}$$

```
B[[2]]
```

```
{2, 3, -1, 2}
```

```
MatrixForm[ A[[ {2,4}, {1,3} ] ] ]
```

$$\begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$$

```
MatrixForm[ A[[ Range[2,4], Range[1,3] ] ] ]
```

$$\begin{pmatrix} 2 & 3 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & 0 \end{pmatrix}$$

```
K = Drop[A, 1];
MatrixForm[K]

$$\begin{pmatrix} 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

```

```
Diagonal[K]
```

```
{2, -1, 0}
```

```
Diagonal[K, 1]
```

```
{3, 1, -5}
```

8.3 Scalars, Vectors, Matrices

Mathematica represents matrices and vectors using lists. Anything that is not a list is considered a scalar.

A vector consists of a list of scalars. A matrix consists of a list of vectors, representing each of its rows; each of these vectors must have the same length.

Operators for testing the structure of vectors and matrices:

VectorQ[<i>expr</i>]	= True = False	<i>expr</i> has the form of a vector otherwise
MatrixQ[<i>expr</i>]	= True = False	<i>expr</i> has the form of a matrix otherwise
Dimensions[<i>expr</i>]		a list of the dimensions of a vector or of a matrix

```
v = {a, b, c} ; VectorQ[ v ]
```

```
True
```

```
VectorQ[ p + q ]
```

```
False
```

```
Di = DiagonalMatrix[{2,1,0,-1,-2}];
```

```
Dimensions[Di]
```

```
{5, 5}
```

```
Dimensions[v]
```

```
{3}
```

```
MatrixQ[ {Di, v} ]
```

```
False
```

```
MatrixQ[ Di ]
```

```
True
```

8.4 Operations on Scalars, Vectors, Matrices

```
Log[ {a, b, c} ]
```

```
{Log[a], Log[b], Log[c]}
```

MatrixForm[Exp[Di]]

$$\begin{pmatrix} e^2 & 1 & 1 & 1 & 1 \\ 1 & e & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \frac{1}{e} & 1 \\ 1 & 1 & 1 & 1 & \frac{1}{e^2} \end{pmatrix}$$

D[{1, x, x^2, x^3}, x]

$$\{0, 1, 2x, 3x^2\}$$

{a, b} + {c, d}

$$\{a + c, b + d\}$$

{a, b, c} + {d, s}

Thread::tdlen : Objects of unequal length in {a, b, c} + {d, s} cannot be combined. >>

$$\{d, s\} + \{a, b, c\}$$

Thread::"tdlen" : Objects of unequal length in {a, b, c} + {d, s} cannot be combined. >>

In general, addition and subtraction can be applied to like lists only. However, a scalar is interpreted as unit list having the same dimensions as the list representing the other operand.

1 + {a, b}

$$\{1 + a, 1 + b\}$$

{a, b} + p

$$\{a + p, b + p\}$$

% /. p -> {c, d}

$$\{\{a + c, a + d\}, \{b + c, b + d\}\}$$

Exp[Di] + c; MatrixForm[%]

$$\begin{pmatrix} a + c & a + d \\ b + c & b + d \end{pmatrix}$$

k {a, b}

$$\{a k, b k\}$$

{a, b}/k

$$\left\{ \frac{a}{k}, \frac{b}{k} \right\}$$

A // MatrixForm

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

Diagonal[A]

$$\{1, 3, 1, -5\}$$

Diagonal[A, 2]

$$\{3, -5\}$$

Diagonal[A, -1]

{2, -1, 0}

D = Drop[A, 1];

MatrixForm[%]

Set::wrsym: Symbol D is Protected. >>

$$\begin{pmatrix} 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

Diagonal[%%]

{2, -1, 0}

Diagonal[%%%, 1]

{3, 1, -5}

8.5 Multiplying Vectors and Matrices

Different kinds of vector and matrix multiplication:

s = scalar; v = vector; m = matrix.

s v, s m multiply each element by a scalar

v.v, v.m, m.v, m.m vector and matrix multiplication

v = {a, b, c}

{a, b, c}

s v

{a s, b s, c s}

v . {ap, bp, cp}

a ap + b bp + c cp

v . v

$a^2 + b^2 + c^2$

{a,b,c} {d, e, f}

{a d, b e, c f}

{{a, b}, {c, d}} . {x, y}

{a x + b y, c x + d y}

m1 = {{a, b}, {c, d}}; m2 = {{1,2}, {3,4}};

MatrixForm[m1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

MatrixForm[m2]

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

m = m1 . m2

{{a + 3 b, 2 a + 4 b}, {c + 3 d, 2 c + 4 d}}

MatrixForm[m]

$$\begin{pmatrix} a + 3 b & 2 a + 4 b \\ c + 3 d & 2 c + 4 d \end{pmatrix}$$

v = {**x**, **y**}

{**x**, **y**}

MatrixForm[**m1** . **v**]

$$\begin{pmatrix} a x + b y \\ c x + d y \end{pmatrix}$$

MatrixForm[**v** . **m1**]

$$\begin{pmatrix} a x + c y \\ b x + d y \end{pmatrix}$$

ExpandAll[**MatrixForm**[**v** . **m1** . **v**]]

$$a x^2 + b x y + c x y + d y^2$$

8.5.1 The outer product

Outer[**Times**, **t1**, **t2**, ...] Outer Product

Outer[**Times**, {**a**, **b**, **c**}, {**x**, **y**, **z**}]

{{**a x**, **a y**, **a z**}, {**b x**, **b y**, **b z**}, {**c x**, **c y**, **c z**}}

MatrixForm[%]

$$\begin{pmatrix} a x & a y & a z \\ b x & b y & b z \\ c x & c y & c z \end{pmatrix}$$

The command **Outer**[**Times**,**v1**,**v2**] does not give the vector product of two vectors but the tensor product.

r = {**x**, **y**, **z**};

IdentityMatrix[3] **r.r** - **Outer**[**Times**, **r**, **r**]/**MatrixForm**

$$\begin{pmatrix} y^2 + z^2 & -x y & -x z \\ -x y & x^2 + z^2 & -y z \\ -x z & -y z & x^2 + y^2 \end{pmatrix}$$

Moment of inertia = second moment = Trägheitstensor

The operator in **Outer**[] need not be **Times**. It may be any function **f**. The lengths of the lists need not be equal.

f[**i**_, **j**_] = **i** / **j**;

Outer[**f**, {**1**, **2**, **3**}, {**2**, **4**, **8**, **10**}]

{{ $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{10}$ }, {1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$ }, { $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{10}$ }}

Outer[**List**, {**1**, **2**, **3**}, {**2**, **4**, **8**, **10**}]

{{{**1**, **2**}, {**1**, **4**}, {**1**, **8**}, {**1**, **10**}},
{**2**, **2**}, {**2**, **4**}, {**2**, **8**}, {**2**, **10**}}, {{**3**, **2**}, {**3**, **4**}, {**3**, **8**}, {**3**, **10**}}}

Outer[**Times**, {**a**, **b**}, {**c**, **d**}, {**e**,**f**}]

{{{**a c e**, **a c f**}, {**a d e**, **a d f**}}, {{**b c e**, **b c f**}, {**b d e**, **b d f**}}}

```

MatrixForm[%]

$$\begin{pmatrix} \begin{pmatrix} a & c & e \\ a & c & f \end{pmatrix} & \begin{pmatrix} a & d & e \\ a & d & f \end{pmatrix} \\ \begin{pmatrix} b & c & e \\ b & c & f \end{pmatrix} & \begin{pmatrix} b & d & e \\ b & d & f \end{pmatrix} \end{pmatrix}$$

Clear[x,y,z]; r = {x, y, z};
m1 = {{a, b}, {c, d}}; m2 = {{1,2}, {3,4}};
m12 = MatrixForm[ Outer[Times, m1, m2] ]

$$\begin{pmatrix} \begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix} & \begin{pmatrix} b & 2b \\ 3b & 4b \end{pmatrix} \\ \begin{pmatrix} c & 2c \\ 3c & 4c \end{pmatrix} & \begin{pmatrix} d & 2d \\ 3d & 4d \end{pmatrix} \end{pmatrix}$$

m21 = MatrixForm[ Outer[Times, m2, m1] ]

$$\begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} \\ \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix} & \begin{pmatrix} 4a & 4b \\ 4c & 4d \end{pmatrix} \end{pmatrix}$$


```

8.5.2 The Kronecker product

The Direct Product (or Kronecker Product) of two matrices is defined as (where there are **no** submatrices !)

$$A \otimes B = \begin{pmatrix} a_{1,1} B & a_{1,2} B & \dots & a_{1,n} B \\ a_{2,1} B & a_{2,2} B & \dots & a_{2,n} B \\ \dots & \dots & \dots & \dots \\ a_{m,1} B & a_{m,2} B & \dots & a_{m,n} B \end{pmatrix}$$

So the Outer Product of *Mathematica* does **not** correspond to the Kronecker product. However, there is a special command **KroneckerProduct** [] :

```

Clear[m1, m2]
m1 = {{a, b}, {c, d}};
m2 = {{1, 2, 3}, {4, 5, 6}};
MatrixForm[m1]
MatrixForm[m2]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$


$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

KroneckerProduct[m1, m2] // MatrixForm

$$\begin{pmatrix} a & 2a & 3a & b & 2b & 3b \\ 4a & 5a & 6a & 4b & 5b & 6b \\ c & 2c & 3c & d & 2d & 3d \\ 4c & 5c & 6c & 4d & 5d & 6d \end{pmatrix}$$


```

8.6 Basic Matrix Operations

Transpose [m]	transpose of m
Inverse [m]	inverse of m
Det [m]	determinant of m
Minors [m, k]	a matrix of the k x k minors of m

MatrixRank[m] rank of the matrix **m**

Tr[m] trace of **m**

MatrixForm[A]

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

MatrixForm[Transpose[A]]

$$\begin{pmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & -1 & 2 \\ 3 & 0 & 1 & 0 \\ 4 & -5 & 1 & -5 \end{pmatrix}$$

Det[A]

135

B = Inverse[A]; MatrixForm[B]

$$\begin{pmatrix} -\frac{1}{27} & \frac{1}{5} & \frac{1}{9} & -\frac{28}{135} \\ \frac{4}{27} & \frac{1}{5} & -\frac{4}{9} & -\frac{23}{135} \\ \frac{4}{27} & -\frac{1}{5} & \frac{5}{9} & \frac{58}{135} \\ \frac{2}{27} & 0 & -\frac{2}{9} & -\frac{5}{27} \end{pmatrix}$$

MatrixForm[A . B]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

G = A; G[[1]] = G[[2]]; MatrixForm[G]

$$\begin{pmatrix} 2 & 3 & 0 & -5 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

So **G** is a singular matrix:

Inverse[G]

Inverse::sing : Matrix {{2, 3, 0, -5}, {2, 3, 0, -5}, {2, -1, 1, 1}, {-2, 2, 0, -5}} is singular. >>

Inverse[{{2, 3, 0, -5}, {2, 3, 0, -5}, {2, -1, 1, 1}, {-2, 2, 0, -5}}]

Inverse::sing : Matrix {{2, 3, 0, -5}, {2, 3, 0, -5}, {2, -1, 1, 1}, {-2, 2, 0, -5}} is singular. >>

Det[G]

0

MatrixRank[G]

3

```
ma = Array[a, {3, 4}]; MatrixForm[ma]
```

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$$

```
mi3 = Minors[ma, 3]
```

$$\left\{ \left\{ -a_{1,3} a_{2,2} a_{3,1} + a_{1,2} a_{2,3} a_{3,1} + a_{1,3} a_{2,1} a_{3,2} - a_{1,1} a_{2,3} a_{3,2} - a_{1,2} a_{2,1} a_{3,3} + a_{1,1} a_{2,2} a_{3,3}, \right. \right. \\ \left. -a_{1,4} a_{2,2} a_{3,1} + a_{1,2} a_{2,4} a_{3,1} + a_{1,4} a_{2,1} a_{3,2} - a_{1,1} a_{2,4} a_{3,2} - a_{1,2} a_{2,1} a_{3,4} + a_{1,1} a_{2,2} a_{3,4}, \right. \\ \left. -a_{1,4} a_{2,3} a_{3,1} + a_{1,3} a_{2,4} a_{3,1} + a_{1,4} a_{2,1} a_{3,3} - a_{1,1} a_{2,4} a_{3,3} - a_{1,3} a_{2,1} a_{3,4} + a_{1,1} a_{2,3} a_{3,4}, \right. \\ \left. -a_{1,4} a_{2,3} a_{3,2} + a_{1,3} a_{2,4} a_{3,2} + a_{1,4} a_{2,2} a_{3,3} - a_{1,2} a_{2,4} a_{3,3} - a_{1,3} a_{2,2} a_{3,4} + a_{1,2} a_{2,3} a_{3,4} \right\} \right\}$$

```
Dimensions[mi3]
```

```
{1, 4}
```

```
mi2 = Minors[ma, 2]
```

$$\left\{ \left\{ -a_{1,2} a_{2,1} + a_{1,1} a_{2,2}, -a_{1,3} a_{2,1} + a_{1,1} a_{2,3}, -a_{1,4} a_{2,1} + a_{1,1} a_{2,4}, \right. \right. \\ \left. -a_{1,3} a_{2,2} + a_{1,2} a_{2,3}, -a_{1,4} a_{2,2} + a_{1,2} a_{2,4}, -a_{1,4} a_{2,3} + a_{1,3} a_{2,4} \right\}, \\ \left\{ -a_{1,2} a_{3,1} + a_{1,1} a_{3,2}, -a_{1,3} a_{3,1} + a_{1,1} a_{3,3}, -a_{1,4} a_{3,1} + a_{1,1} a_{3,4}, \right. \\ \left. -a_{1,3} a_{3,2} + a_{1,2} a_{3,3}, -a_{1,4} a_{3,2} + a_{1,2} a_{3,4}, -a_{1,4} a_{3,3} + a_{1,3} a_{3,4} \right\}, \\ \left\{ -a_{2,2} a_{3,1} + a_{2,1} a_{3,2}, -a_{2,3} a_{3,1} + a_{2,1} a_{3,3}, -a_{2,4} a_{3,1} + a_{2,1} a_{3,4}, \right. \\ \left. -a_{2,3} a_{3,2} + a_{2,2} a_{3,3}, -a_{2,4} a_{3,2} + a_{2,2} a_{3,4}, -a_{2,4} a_{3,3} + a_{2,3} a_{3,4} \right\} \right\}$$

```
Dimensions[mi2]
```

```
{3, 6}
```

```
mi1 = Minors[ma, 1] // MatrixForm
```

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$$

```
Minors[G, 3]
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {-10, 20, -20, -5}, {-10, 20, -20, -5}}
```

So the matrix **G** has rank 3.

```
Tr[A]
```

```
0
```

```
Tr[G]
```

```
1
```

8.6.1 Characteristic Polynomial of a Matrix **A** :

```
A = {{1,2,3,4}, {2,3,0,-5}, {2,-1,1,1},{-2,2,0,-5}};
```

```
f = Det[A - x IdentityMatrix[4]]
```

```
135 - 72 x - 10 x^2 + x^4
```

```
cp = CharacteristicPolynomial[A, λ]
```

```
135 - 72 λ - 10 λ^2 + λ^4
```

```
Solve[f == 0., x]
```

```
{{x → -3.00593 - 3.16604 i}, {x → -3.00593 + 3.16604 i}, {x → 1.60861}, {x → 4.40326}}
```

8.6.2 Functions of a Matrix **A**

MatrixPower[m, n]	n -th power of matrix m
--------------------------	--------------------------------

MatrixExp[m]	matrix exponential
---------------------	--------------------

```
Clear[a, b, c, d]
```

```
m = {{a, b}, {c, d}}; MatrixForm[m]
```

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

```
MatrixForm[ m^2 ]
```

$$\begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$$

```
MatrixForm[ m . m ]
```

$$\begin{pmatrix} a^2 + b c & a b + b d \\ a c + c d & b c + d^2 \end{pmatrix}$$

```
MatrixForm[ MatrixPower[m, 2] ]
```

$$\begin{pmatrix} a^2 + b c & a b + b d \\ a c + c d & b c + d^2 \end{pmatrix}$$

```
dd = DiagonalMatrix[{1, -1}]; MatrixForm[dd]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
MatrixExp[dd]//MatrixForm
```

$$\begin{pmatrix} e & 0 \\ 0 & \frac{1}{e} \end{pmatrix}$$

```
dd = I x DiagonalMatrix[{1, -1}]; MatrixForm[dd]
```

$$\begin{pmatrix} i x & 0 \\ 0 & -i x \end{pmatrix}$$

```
MatrixExp[dd]
```

$$\left\{ \left\{ e^{i x}, 0 \right\}, \left\{ 0, e^{-i x} \right\} \right\}$$

```
ComplexExpand[%] // MatrixForm
```

$$\begin{pmatrix} \cos[x] + i \sin[x] & 0 \\ 0 & \cos[x] - i \sin[x] \end{pmatrix}$$

MatrixExp[m]

$$\left\{ \left\{ -\frac{\left(a - d - \sqrt{a^2 + 4bc - 2ad + d^2} \right) e^{\frac{a}{2} + \frac{d}{2} - \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{2 \sqrt{a^2 + 4bc - 2ad + d^2}} + \frac{\left(a - d + \sqrt{a^2 + 4bc - 2ad + d^2} \right) e^{\frac{a}{2} + \frac{d}{2} + \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{2 \sqrt{a^2 + 4bc - 2ad + d^2}}, \right. \right.$$

$$\left. -\frac{b e^{\frac{a}{2} + \frac{d}{2} - \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{\sqrt{a^2 + 4bc - 2ad + d^2}} + \frac{b e^{\frac{a}{2} + \frac{d}{2} + \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{\sqrt{a^2 + 4bc - 2ad + d^2}} \right\},$$

$$\left\{ -\frac{c e^{\frac{a}{2} + \frac{d}{2} - \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{\sqrt{a^2 + 4bc - 2ad + d^2}} + \frac{c e^{\frac{a}{2} + \frac{d}{2} + \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{\sqrt{a^2 + 4bc - 2ad + d^2}}, \right.$$

$$\left. -\frac{\left(-a + d - \sqrt{a^2 + 4bc - 2ad + d^2} \right) e^{\frac{a}{2} + \frac{d}{2} - \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{2 \sqrt{a^2 + 4bc - 2ad + d^2}} + \frac{\left(-a + d + \sqrt{a^2 + 4bc - 2ad + d^2} \right) e^{\frac{a}{2} + \frac{d}{2} + \frac{1}{2} \sqrt{a^2 + 4bc - 2ad + d^2}}}{2 \sqrt{a^2 + 4bc - 2ad + d^2}} \right\} \left. \right\}$$

FullSimplify[%]

$$\left\{ \left\{ \frac{e^{\frac{a+d}{2}} \left(\sqrt{4bc + (a-d)^2} \operatorname{Cosh} \left[\frac{1}{2} \sqrt{4bc + (a-d)^2} \right] + (a-d) \operatorname{Sinh} \left[\frac{1}{2} \sqrt{4bc + (a-d)^2} \right] \right)}{\sqrt{4bc + (a-d)^2}}, \right. \right.$$

$$\frac{2b e^{\frac{a+d}{2}} \operatorname{Sinh} \left[\frac{1}{2} \sqrt{4bc + (a-d)^2} \right]}{\sqrt{4bc + (a-d)^2}} \left. \right\}, \left\{ \frac{2c e^{\frac{a+d}{2}} \operatorname{Sinh} \left[\frac{1}{2} \sqrt{4bc + (a-d)^2} \right]}{\sqrt{4bc + (a-d)^2}}, \right.$$

$$\left. \frac{e^{\frac{a+d}{2}} \left(\sqrt{4bc + (a-d)^2} \operatorname{Cosh} \left[\frac{1}{2} \sqrt{4bc + (a-d)^2} \right] + (-a+d) \operatorname{Sinh} \left[\frac{1}{2} \sqrt{4bc + (a-d)^2} \right] \right)}{\sqrt{4bc + (a-d)^2}} \right\} \left. \right\}$$

8.7 Solving Linear Systems

Systems of linear equations can be solved by various *Mathematica* commands.

m = matrix, **x** = vector (unknowns), **b** = vector (right hand side of an equation) .

$$\mathbf{m.x == b}$$

Solve[eq, unk] Solves the system of linear equations *eq* for the unknowns contained in list *unk*. Works for regular and singular systems.

LinearSolve[m, b] gives a vector **x** which solves the matrix equation **m.x == b**

NullSpace[m] a list of basis vectors whose linear combinations **x** fulfil the matrix equation **m.x == 0**

RowReduce[m] a simplified form of **m** obtained by making linear combinations of rows

Let n denote the number of unknowns = number of columns of the matrix \mathbf{m} .

r is the rank of the matrix (command `MatrixRank[]`, § 8.6).

If \mathbf{m} is a square matrix and is regular, $n = r$, then the system of equations has a unique solution, in particular $\mathbf{x} = 0$ if $\mathbf{b} = 0$.

If $r < n$ the homogeneous system does not have a unique solution; in fact, there is a solution manifold depending on $n - r$ arbitrary parameters. The command `NullSpace[]` gives a basis for that manifold.

If $r > n$, there is no exact solution; an approximate solution may be found by optimization methods.

The inhomogeneous system with $r < n$ has solutions only if the vector \mathbf{b} on the right hand side does not introduce a contradiction.

This implies: the rank of the matrix formed by adjoining \mathbf{b} to \mathbf{m} is equal to the rank of \mathbf{m} ; this condition is equivalent to the following one: all vectors of the nullspace of the matrix $\hat{\mathbf{m}}$ (= matrix transposed to \mathbf{m}) are orthogonal to \mathbf{b} .

```
m = {{1, 5}, {2, 1}}; MatrixForm[m]
```

$$\begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}$$

The commands listed above provide several methods for solving systems of linear equations.

8.7.1 Method 1 uses the command `Solve[]`.

```
m . {x, y} == {a, b}
```

```
{x + 5 y, 2 x + y} == {a, b}
```

```
Solve[% , {x, y}]
```

$$\left\{ \left\{ \mathbf{x} \rightarrow \frac{1}{9} (-\mathbf{a} + 5 \mathbf{b}), \mathbf{y} \rightarrow \frac{1}{9} (2 \mathbf{a} - \mathbf{b}) \right\} \right\}$$

`Solve[]` works also for solvable singular linear systems:

```
{x + y, 2x + 2y} == {0,0};
```

```
Solve[% , {x, y}]
```

`Solve::svars` : Equations may not give solutions for all "solve" variables. >>

```
{{y → -x}}
```

`Solve::"svars"` : Equations may not give solutions for all "solve" variables. >>

The following singular system is contradictory; thus no solution exists:

```
{x + y, 2x + 2y} == {a,b};
```

```
Solve[% , {x, y}]
```

```
{}
```

8.7.2 Method 2 for solving a regular system uses the inverse matrix :

```
Inverse[m] . {a, b}
```

$$\left\{ -\frac{\mathbf{a}}{9} + \frac{5 \mathbf{b}}{9}, \frac{2 \mathbf{a}}{9} - \frac{\mathbf{b}}{9} \right\}$$

8.7.3 In method 3 one works with the command `LinearSolve[]`.

```
LinearSolve[m, {a, b} ]
```

$$\left\{ \frac{1}{9} (-\mathbf{a} + 5 \mathbf{b}), \frac{1}{9} (2 \mathbf{a} - \mathbf{b}) \right\}$$

`LinearSolve[]` does not give the general solution of a solvable singular system, but only a special solution.

```
LinearSolve[{{1,1},{2,2}}, {0,0} ]
```

```
{0, 0}
```

For large systems methods 3 and 4 work faster than the first two methods.

8.7.4 In method 4 one uses the command `RowReduce[]`.

For that purpose the vector on the right hand side must be joined to the matrix of coefficients to give a rectangular matrix:

```
h1 = Append[m[[1]], a];
h2 = Append[m[[2]], b];
ma = {h1,h2}; MatrixForm[ma]
```

$$\begin{pmatrix} 1 & 5 & a \\ 2 & 1 & b \end{pmatrix}$$

```
RowReduce[ma] //MatrixForm
```

$$\begin{pmatrix} 1 & 0 & \frac{1}{9}(-a + 5b) \\ 0 & 1 & \frac{1}{9}(2a - b) \end{pmatrix}$$

The last column of the above matrix contains the solutions of the linear system.

```
mm = {{1,1,0}, {2,2,0}}; MatrixForm[mm]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

```
RowReduce[mm] //MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
mm = {{1,1,a}, {2,2,b}}; MatrixForm[mm]
```

$$\begin{pmatrix} 1 & 1 & a \\ 2 & 2 & b \end{pmatrix}$$

```
RowReduce[mm] //MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This shows again that the singular system with the rhs $\{\mathbf{a}, \mathbf{b}\}$ is contradictory.

Method 4 is shown again for the following system:

$$\begin{aligned} -3x + 2y + 11z &= 1, \\ x + 3y + 7z &= -5, \\ -2x - 3y + 5z &= 2. \end{aligned}$$

```
g = {{-3,2,11,1},{1,3,7,-5},{-2,-3,5,2}}; MatrixForm[g]
```

$$\begin{pmatrix} -3 & 2 & 11 & 1 \\ 1 & 3 & 7 & -5 \\ -2 & -3 & 5 & 2 \end{pmatrix}$$

```
rr = RowReduce[g]; MatrixForm[rr]
```

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{213}{113} \\ 0 & 1 & 0 & -\frac{10}{113} \\ 0 & 0 & 1 & -\frac{46}{113} \end{pmatrix}$$

```
v = Transpose[rr][[4]]
```

$$\left\{ -\frac{213}{113}, -\frac{10}{113}, -\frac{46}{113} \right\}$$

This is the solution; as is verified below:

```
cm = g[[ Range[3], Range[3] ]]; MatrixForm[cm]
```

$$\begin{pmatrix} -3 & 2 & 11 \\ 1 & 3 & 7 \\ -2 & -3 & 5 \end{pmatrix}$$

```
cm . v == Transpose[g][[4]]
```

```
True
```

```
me = {{1,1,1,-1}, {1,2,3,-4}, {1,3,6,-10},{1,4,10,-a}}
{{1, 1, 1, -1}, {1, 2, 3, -4}, {1, 3, 6, -10}, {1, 4, 10, -a}}
```

```
me[[Range[4],Range[3]]] . {x,y,z} == me[[4]]//Thread
```

```
{x+y+z == 1, x+2 y+3 z == 4, x+3 y+6 z == 10, x+4 y+10 z == -a}
```

```
Solve[%, {x,y,z}]
```

```
{}
```

```
me[[ Range[3],Range[3] ]] . {x,y,z} == me[[ 4,Range[3] ]]/Thread
```

```
{x+y+z == 1, x+2 y+3 z == 4, x+3 y+6 z == 10}
```

```
Solve[%, {x,y,z}] //Flatten
```

```
{x → 1, y → -3, z → 3}
```

So the system of 4 equations above is overdetermined. But the system comprising the first 3 equations is solvable. This is also verified using **RowReduce[]**.

```
met = RowReduce[me]; MatrixForm[met]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
mer = RowReduce[ me[[Range[3],Range[3] ]] ]; MatrixForm[mer]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we look for the value of a for which the system of 4 equations is solvable:

```
sa = Solve[Det[me] == 0, a]//Flatten
```

```
{a → 19}
```

```
RowReduce[me /. sa]//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Solve[(me /. sa) . {x,y,z,1} == 0, {x,y,z}]
```

```
{{x → 1, y → -3, z → 3}}
```

8.7.5 Eliminate

```
?? Eliminate
```

Eliminate[eqns, vars] eliminates variables between a set of simultaneous equations. >>

Attributes[Eliminate] = {Protected}

Options[Eliminate] = {InverseFunctions → Automatic, MakeRules → False, Method → 1, Mode → Generic, Sort → True, VerifySolutions → Automatic, WorkingPrecision → ∞}

eqs = {2 x + 3 y + z == 1, x - y - z == 4, 3 x + 7 z == 5};

Solve[eqs, {x, y, z}] // Flatten

$\left\{x \rightarrow \frac{101}{41}, y \rightarrow -\frac{49}{41}, z \rightarrow -\frac{14}{41}\right\}$

e1 = Eliminate[eqs, z]

11 x == 2 - 21 y && 41 y == -49

Apply[List, e1]

{11 x == 2 - 21 y, 41 y == -49}

Solve[%, {x, y}] // Flatten

$\left\{x \rightarrow \frac{101}{41}, y \rightarrow -\frac{49}{41}\right\}$

8.7.6 The Null Space

A regular matrix has no nullspace.

A = {{1,2,3,4}, {2,3,0,-5}, {2,-1,1,1},{-2,2,0,-5}};
MatrixForm[A]

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ -2 & 2 & 0 & -5 \end{pmatrix}$$

Det[A]

135

NullSpace[A]

{}

Now a singular matrix is constructed:

G = A; G[[4]] = G[[1]]; MatrixForm[G]

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

rg = MatrixRank[G]

3

Length[G] - rg

1

v = NullSpace[G]//Flatten

{28, 23, -58, 25}

G . v

{0, 0, 0, 0}

v . G

{-17, 233, 101, 39}

v.Transpose[G]

{0, 0, 0, 0}

x = {x1, x2, x3, x4}

{x1, x2, x3, x4}

so = Solve[G.x == 0, x] // Flatten

Solve::svars : Equations may not give solutions for all "solve" variables. >>

$$\left\{ x_2 \rightarrow \frac{23 x_1}{28}, x_3 \rightarrow -\frac{29 x_1}{14}, x_4 \rightarrow \frac{25 x_1}{28} \right\}$$

Solve::svars : Equations may not give solutions for all "solve" variables. >>

xs = x /. so /. x4 -> 25 a

$$\left\{ x_1, \frac{23 x_1}{28}, -\frac{29 x_1}{14}, \frac{25 x_1}{28} \right\}$$

v a

{28 a, 23 a, -58 a, 25 a}

G.xs

{0, 0, 0, 0}

A new singular matrix is constructed:

H = G; H[[3]] = 2 G[[1]]; MatrixForm[H]

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & -5 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

rh = MatrixRank[H]

2

Length[H] - rh

2

so = Solve[H.x == 0, x]

Solve::svars : Equations may not give solutions for all "solve" variables. >>

$$\left\{ \left\{ x_3 \rightarrow -\frac{13 x_1}{15} - \frac{22 x_2}{15}, x_4 \rightarrow \frac{2 x_1}{5} + \frac{3 x_2}{5} \right\} \right\}$$

Solve::svars : Equations may not give solutions for all "solve" variables. >>

ns = NullSpace[H]

{{22, -13, 0, 1}, {9, -6, 1, 0}}

ys = a ns[[1]] + b ns[[2]]

{22 a + 9 b, -13 a - 6 b, b, a}

```
H.y // Together
```

```
{0, 0, 0, 0}
```

Now a soluble inhomogeneous singular system $\mathbf{H}\cdot\mathbf{x} = \mathbf{b}$ is constructed.

```
b = {b1, b2, b3, b4}
```

```
{b1, b2, b3, b4}
```

This vector \mathbf{b} must be orthogonal to all solutions of the homogeneous system transposed to \mathbf{H} :

```
nt = NullSpace[Transpose[H]]
```

```
{{-1, 0, 0, 1}, {-2, 0, 1, 0}}
```

```
sys = {nt[[1]].b, nt[[2]].b}
```

```
{-b1 + b4, -2 b1 + b3}
```

```
so = Solve[sys == 0, b] // Flatten
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

```
{b3 → 2 b1, b4 → b1}
```

```
vb = b /. so
```

```
{b1, b2, 2 b1, b1}
```

Only such vectors on the right hand side of $\mathbf{H}\cdot\mathbf{x} = \mathbf{b}$ lead to a non-trivial solution:

```
Hb = Append[Transpose[H], vb /. so] // Transpose; MatrixForm[Hb]
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 & b1 \\ 2 & 3 & 0 & -5 & b2 \\ 2 & 4 & 6 & 8 & 2 b1 \\ 1 & 2 & 3 & 4 & b1 \end{pmatrix}$$

```
MatrixRank[H]
```

```
MatrixRank[Hb]
```

```
2
```

```
2
```

8.8 Eigenvalues and Eigenvectors. Jordan Decomposition

An eigenvector X of a square $n \times n$ matrix A is a vector whom the application of the matrix transforms into a collinear vector αX such that

$$AX = \alpha X. \quad (1)$$

Any vector fulfilling this condition is called an eigenvector X_i ; the corresponding value α is called the eigenvalue α_i .

The above equation is rewritten by use of the identity matrix E :

$$(A - \alpha E)X = 0. \quad (2)$$

This linear system for the components of the vector X has a nontrivial solution only if the matrix of the coefficients is singular. The condition that its determinant must be zero, gives a polynomial of order n in α , the characteristic polynomial $P(\alpha)$, the characteristic equation $P(\alpha) = 0$ respectively:

$$P(\alpha) := \det(A - \alpha E) = (-1)^n [\alpha^n - (a_{11} + a_{22} + \dots + a_{nn}) \alpha^{n-1} + \dots + (-1)^n \det(A)] = 0. \quad (3)$$

Its zeros are the eigenvalues α_i . After an eigenvalue α_i has been inserted for α in eq.(2) one can solve it to find the corresponding eigenvector X_i .

A matrix M may be transformed by an equivalence transformation. Such a transformation is performed with the help of a regular matrix T such that

$$B = T^{-1} A T.$$

If the matrix T consists of the eigenvectors X_i of the matrix A , then the matrix B becomes a diagonal matrix, i.e.

$$T^{-1} A T = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n).$$

Multiple eigenvalues are called degenerate. Then it may happen that the number of independent eigenvectors is smaller than the dimension of the matrix A . In this case it is impossible to build the matrix T and to transform the Matrix A into a purely diagonal form. But, it can be transformed to Jordan normal form (cf. next section).

The following types of matrices always can be diagonalized:

1. Matrices whose eigenvalues are pairwise different, i.e. all the eigenvalues are non-degenerate.
2. Normal matrices; they fulfil the following condition: $AA^+ = A^+A$, where $A^+ = \tilde{A}^*$ is the complex conjugate transpose to A .

2.1 Real ($A = A^*$) normal matrices are the following ones:

- 2.1.1 symmetric ($A = \tilde{A}$), 2.1.2 antisymmetric ($A = -\tilde{A}$), 2.1.3 orthogonal ($A\tilde{A} = E$),
- 2.1.4 anti-orthogonal ($A\tilde{A} = -E$);

2.2 Complex normal matrices are the following ones:

- 2.2.1 Hermitian ($A = A^+$), 2.2.2 Anti-Hermitian ($A = -A^+$), 2.2.3 unitary ($AA^+ = E$),
- 2.2.4 antiunitary ($AA^+ = -E$),

The eigenvectors of real matrices of type 2.1.1 and 2.1.2 can be chosen to be real and orthonormal; then the matrix T is orthogonal. In all other cases the eigenvectors can be chosen to be complex orthonormal; then the matrix T is unitary.

8.8.1 Eigenvalue Decomposition of a Matrix

Eigenvalues [m]	a list of the eigenvalues of m
Eigenvectors [m]	a list of the eigenvectors of m
Eigensystem [m]	a list of the form {eigenvalues, -vectors}
Eigenvalues [$N[m]$]	numerical eigenvalues
Eigenvalues [$N[m,k]$]	numerical eigenvalues, starting with k -digit precision

Clear[x]

A = {{1,1,2}, {1,2,1}, {2,1,1}}; MatrixForm[A]

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Eigenvalues[**A**]

{4, -1, 1}

The characteristic equation **ceq** of the matrix is found below:

AM = A - x IdentityMatrix[3]; MatrixForm[AM]

$$\begin{pmatrix} 1-x & 1 & 2 \\ 1 & 2-x & 1 \\ 2 & 1 & 1-x \end{pmatrix}$$

```

ceq = Det[AM]
-4 + x + 4 x2 - x3

ds = Solve[ceq == 0, x]
{{x → -1}, {x → 1}, {x → 4}}

Eigenvectors[A]
{{1, 1, 1}, {-1, 0, 1}, {1, -2, 1}}

Eigensystem[A]
{{4, -1, 1}, {{1, 1, 1}, {-1, 0, 1}, {1, -2, 1}}}

          eigenvalues      vector1      vector2      vector3

AM1 = AM /. ds[[1]]
{{2, 1, 2}, {1, 3, 1}, {2, 1, 2}}

NullSpace[AM1]
{{-1, 0, 1}}

AM /. ds
{{{2, 1, 2}, {1, 3, 1}, {2, 1, 2}},
 {{0, 1, 2}, {1, 1, 1}, {2, 1, 0}}, {{-3, 1, 2}, {1, -2, 1}, {2, 1, -3}}}

NullSpace[ %[2]] ]
{{1, -2, 1}}

Clear[a, b, c, d, m]
m = {{a, b}, {c, d}}; MatrixForm[m]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$


Eigenvalues[m]

$$\left\{ \frac{1}{2} \left( a + d - \sqrt{a^2 + 4bc - 2ad + d^2} \right), \frac{1}{2} \left( a + d + \sqrt{a^2 + 4bc - 2ad + d^2} \right) \right\}$$


Eigensystem[m]

$$\left\{ \left\{ \frac{1}{2} \left( a + d - \sqrt{a^2 + 4bc - 2ad + d^2} \right), \frac{1}{2} \left( a + d + \sqrt{a^2 + 4bc - 2ad + d^2} \right) \right\}, \right.$$


$$\left. \left\{ \left\{ -\frac{-a + d + \sqrt{a^2 + 4bc - 2ad + d^2}}{2c}, 1 \right\}, \left\{ -\frac{-a + d - \sqrt{a^2 + 4bc - 2ad + d^2}}{2c}, 1 \right\} \right\} \right\}$$


```

8.8.2 Transforming a Matrix to Jordan Form

Not every matrix can be diagonalized. see the discussion preceding the previous subsection. Matrices with multiple eigenvalues may have less eigenvectors than eigenvalues. These may be transformed to a normal form with 1's in the diagonal above the main diagonal containing the degenerate eigenvalues. The `JordanDecomposition[]` applied to a diagonalizable matrix yields the same diagonal matrix as the eigenvalue decomposition.

`JordanDecomposition[m]` yields the Jordan decomposition of a square matrix `m`. The result is a list `{s, j}` where `s` is a similarity matrix and `j` is the Jordan canonical form of `m`.

```
mm = {{1,0,0},{2,1,0},{0,0,-1}}; MatrixForm[mm]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
Eigensystem[mm]
```

```
{{-1, 1, 1}, {{0, 0, 1}, {0, 1, 0}, {0, 0, 0}}}
```

Note the last vector in the list above is a Null vector.

```
jd = JordanDecomposition[mm]
```

```
{{{0, 0, 1/2}, {0, 1, 0}, {1, 0, 0}}, {{-1, 0, 0}, {0, 1, 1}, {0, 0, 1}}}
```

```
MatrixForm[jd[[2]]]
```

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
J = Inverse[jd[[1]]].mm.jd[[1]]; MatrixForm[J]
```

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

8.9 Singular Value Decomposition. PseudoInverse

In this paragraph generalizations are treated which are analogous to eigenvalue decomposition and inverse matrices, but for rectangular and even singular matrices.

The singular value decomposition is applicable to rectangular matrices; it is a procedure which somewhat (but not exactly) resembles the diagonalization (= eigendecomposition) of a square matrix. In fact:

Any real $m \times n$ (with $m \geq n$) matrix A has a representation as $A = \tilde{u} w v$ where $\tilde{u}u = E$, $\tilde{v}v = E$ and w is a diagonal matrix with non-negative entries. The matrix u is $m \times m$, v is $n \times n$ and w is $m \times n$.

The singular values w_i are the square roots of the eigenvalues of $\tilde{A}A$, and the columns of v are the eigenvectors of $\tilde{A}A$. The columns of u are the eigenvectors of $A\tilde{A}$.

Computation of the singular value decomposition is more stable than forming $\tilde{A}A$ and computing the eigendecomposition.

8.9.1 Singular Value Decomposition

SingularValueDecomposition[m]

SingularValues[m] gives the singular value decomposition for a numerical matrix m .

The result is a list $\{u, w, v\}$, where w is the list of singular values, and m can be written as $m = u.w.Conjugate[Transpose[v]]$; or as $m = Conjugate[Transpose[u]].w.v$.

SingularValueList[m] gives a list containing the singular values of the matrix m .

This method is applicable to rectangular matrices, too. Even if the matrix m is square and diagonalizable, the singular values contained in w differ from the eigenvalues.

If u (v) is square, it is orthogonal (or unitary if the original matrix m is complex); otherwise it is orthogonal (or unitary) in one direction only.

Example 1

A = {{1,1,2}, {1,2,1}, {2,1,1}}; MatrixForm[A]

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

SingularValueList[A]

{4, 1, 1}

The singular values contained in the diagonal matrix **w** shown in the above list differ from the eigenvalues of **A** given below:

Eigenvalues[A]

{4, -1, 1}

{u, w, v} = SingularValueDecomposition[A]

$$\left\{ \left\{ \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}} \right\}, \left\{ \frac{1}{\sqrt{3}}, 0, \sqrt{\frac{2}{3}} \right\}, \left\{ \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}} \right\} \right\}, \right. \\ \left. \left\{ \{4, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\} \right\}, \right. \\ \left. \left\{ \left\{ \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}} \right\}, \left\{ \frac{1}{\sqrt{3}}, 0, \sqrt{\frac{2}{3}} \right\}, \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}} \right\} \right\} \right\}$$

MatrixForm[u]

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

MatrixForm[w]

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

MatrixForm[v]

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

u.w.Transpose[v] // MatrixForm

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

This is the original matrix.

Transpose[u].w.v // MatrixForm

$$\begin{pmatrix} 2 & -2\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3} \\ 2\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{6}} & -\frac{5}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3} & \frac{\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix}$$

Example 2

A = {{1,1,2}, {1,I,1}, {2,1,1}}; MatrixForm[A]

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & i & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

{u, w, v} = SingularValueDecomposition[A]

$$\left\{ \left\{ \left\{ \frac{9 - \frac{i}{2}}{\sqrt{195}}, \frac{1}{\sqrt{2}}, \frac{\frac{1}{2} - \frac{3i}{2}}{\sqrt{30}} \right\}, \left\{ \frac{\frac{11}{2} + \frac{3i}{2}}{\sqrt{195}}, 0, -\frac{3 - 4i}{\sqrt{30}} \right\}, \left\{ \frac{9 - \frac{i}{2}}{\sqrt{195}}, -\frac{1}{\sqrt{2}}, \frac{\frac{1}{2} - \frac{3i}{2}}{\sqrt{30}} \right\} \right\}, \right. \\ \left. \left\{ \left\{ \sqrt{13}, 0, 0 \right\}, \left\{ 0, 1, 0 \right\}, \left\{ 0, 0, 1 \right\} \right\}, \right. \\ \left. \left\{ \left\{ \frac{\sqrt{\frac{5}{3}}}{2}, -\frac{1}{\sqrt{2}}, -\frac{\frac{3}{2} + \frac{i}{2}}{\sqrt{30}} \right\}, \left\{ \frac{\frac{3}{2} - \frac{i}{2}}{\sqrt{15}}, 0, \sqrt{\frac{5}{6}} \right\}, \left\{ \frac{\sqrt{\frac{5}{3}}}{2}, \frac{1}{\sqrt{2}}, -\frac{\frac{3}{2} + \frac{i}{2}}{\sqrt{30}} \right\} \right\} \right\}$$

w

$$\left\{ \left\{ \sqrt{13}, 0, 0 \right\}, \left\{ 0, 1, 0 \right\}, \left\{ 0, 0, 1 \right\} \right\}$$

N[w]

$$\left\{ \{3.60555, 0., 0.\}, \{0., 1., 0.\}, \{0., 0., 1.\} \right\}$$

Eigenvalues[A]

$$\left\{ \frac{1}{2} \left((3 + i) + \sqrt{16 - 6i} \right), \frac{1}{2} \left((3 + i) - \sqrt{16 - 6i} \right), -1 \right\}$$

N[%]

$$\{3.53372 + 0.131217 i, -0.533716 + 0.868783 i, -1.\}$$

A == u.w.Conjugate[Transpose[v]] // Simplify

True

Example 3: The singular value decomposition of a rectangular matrix

ma = {{1, -2}, {2, -1}, {1, 1}}; MatrixForm[ma]

$$\begin{pmatrix} 1 & -2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix}$$

```
{u, w, v} = SingularValueDecomposition[ma] // Chop
```

$$\left\{ \left\{ \left\{ -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}} \right\}, \left\{ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{3}} \right\}, \left\{ 0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}} \right\} \right\}, \right. \\ \left. \left\{ \{3, 0\}, \{0, \sqrt{3}\}, \{0, 0\} \right\}, \left\{ \left\{ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}, \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\} \right\} \right\}$$

```
MatrixForm[w]
```

$$\begin{pmatrix} 3 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

```
u.w.Transpose[v] == ma
```

```
True
```

Example 4

```
ma = {{1, 0, 0}, {2, 1, 0}, {0, 0, -1}}; MatrixForm[ma]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
{u, w, v} = SingularValueDecomposition[ma];
```

```
MatrixForm[u]
```

$$\begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{(1+\sqrt{2})^2+(1+2(1+\sqrt{2}))^2}} & 0 & \frac{1-\sqrt{2}}{\sqrt{(1-\sqrt{2})^2+(1+2(1-\sqrt{2}))^2}} \\ \frac{1+2(1+\sqrt{2})}{\sqrt{(1+\sqrt{2})^2+(1+2(1+\sqrt{2}))^2}} & 0 & \frac{1+2(1-\sqrt{2})}{\sqrt{(1-\sqrt{2})^2+(1+2(1-\sqrt{2}))^2}} \\ 0 & -1 & 0 \end{pmatrix}$$

```
MatrixForm[w]
```

$$\begin{pmatrix} \sqrt{3+2\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{3-2\sqrt{2}} \end{pmatrix}$$

```
MatrixForm[v]
```

$$\begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{1+(1+\sqrt{2})^2}} & 0 & \frac{1-\sqrt{2}}{\sqrt{1+(1-\sqrt{2})^2}} \\ \frac{1}{\sqrt{1+(1+\sqrt{2})^2}} & 0 & \frac{1}{\sqrt{1+(1-\sqrt{2})^2}} \\ 0 & 1 & 0 \end{pmatrix}$$

```
MatrixForm[u.w.Transpose[v]] // FullSimplify
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In this particular case the elements of all three matrices may be expressed by symbolic expressions.

$$\mathbf{wa} = \{\sqrt{2} + 1, 1, \sqrt{2} - 1\};$$

N[wa] - w // Chop

```
{0, 2.41421, 2.41421}, {1., 0, 1.}, {0.414214, 0.414214, 0}}
```

Eigenvalues[ma]

```
{-1, 1, 1}
```

Compare these eigenvalues to the elements of **w** or **wa**. The elements of the transformation matrix are:

$$\mathbf{v1} = \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}} \quad (* = \text{Sin}[\pi/8] *);$$

$$\mathbf{v2} = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \quad (* = \text{Cos}[\pi/8] *);$$

ua = {{v1, 0, -v2}, {v2, 0, v1}, {0, -1, 0}}; MatrixForm[ua]

$$\begin{pmatrix} \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}} & 0 & -\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \\ \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} & 0 & \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}} \\ 0 & -1 & 0 \end{pmatrix}$$

ua // N

u // N

```
{{0.382683, 0., -0.92388}, {0.92388, 0., 0.382683}, {0., -1., 0.}}
```

```
{{0.382683, 0., -0.92388}, {0.92388, 0., 0.382683}, {0., -1., 0.}}
```

va = {{v2, 0, -v1}, {v1, 0, v2}, {0, 1, 0}}; MatrixForm[va]

$$\begin{pmatrix} \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} & 0 & -\sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}} \\ \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}} & 0 & \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \\ 0 & 1 & 0 \end{pmatrix}$$

va // N

v // N

```
{{0.92388, 0., -0.382683}, {0.382683, 0., 0.92388}, {0., 1., 0.}}
```

```
{{0.92388, 0., -0.382683}, {0.382683, 0., 0.92388}, {0., 1., 0.}}
```

Example 5

ma = {{1, 0, 0, 3}, {2, 1, 0, 1}, {0, 0, -1, 1}}; MatrixForm[ma]

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

{u, w, v} = SingularValueDecomposition[ma];

MatrixForm[w] // N

$$\begin{pmatrix} 3.76192 & 0. & 0. & 0. \\ 0. & 1.73205 & 0. & 0. \\ 0. & 0. & 0.920833 & 0. \end{pmatrix}$$

```

MatrixForm[u] // N
MatrixForm[v] // N
  ( 0.812989  0.408248  -0.415189 )
  ( 0.528596  -0.816497  0.232206 )
  ( 0.244202  0.408248  0.879601 )

  ( 0.497134  -0.707107  0.0534558  0.5 )
  ( 0.140512  -0.471405  0.25217  -0.833333 )
  ( -0.0649142  -0.235702  -0.955224  -0.166667 )
  ( 0.853756  0.471405  -0.145258  -0.166667 )

```

```

MatrixForm[u.w.Transpose[v] - ma // N // Chop]
  ( 0 0 0 0 )
  ( 0 0 0 0 )
  ( 0 0 0 0 )

```

```

Transpose[u].u // N // Chop // MatrixForm
  ( 1.  0  0 )
  ( 0  1.  0 )
  ( 0  0  1. )

```

```

u.Transpose[u] // N // Chop // MatrixForm
  ( 1.  0  0 )
  ( 0  1.  0 )
  ( 0  0  1. )

```

```

v.Transpose[v] // N // Chop // MatrixForm
  ( 1.  0  0  0 )
  ( 0  1.  0  0 )
  ( 0  0  1.  0 )
  ( 0  0  0  1. )

```

```

Transpose[v].v // N // Chop // MatrixForm
  ( 1.  0  0  0 )
  ( 0  1.  0  0 )
  ( 0  0  1.  0 )
  ( 0  0  0  1. )

```

8.9.2 Pseudoinverse

The inverse A^{-1} of a matrix A exists only if A is square and has full rank. In this case, $Ax = b$ has the solution $x = A^{-1}b$. The pseudoinverse \bar{A} is a generalization of the inverse, and exists for any $m \times n$ matrix. We assume $m > n$. If A has full rank (n) we define:

$$\bar{A} = (\tilde{A} A)^{-1} \tilde{A} \quad (1)$$

with \tilde{A} the transpose to A , and the solution of $Ax = b$ is

$$x = \bar{A} b. \quad (2)$$

The best way to compute \bar{A} is to use singular value decomposition.

If $W = (w_{ij})$ is a diagonal matrix with the entries w_i , then we define the pseudoinverse \bar{W} as:

$$\begin{aligned} \bar{W} = (\bar{w}_{ij}) &= \frac{1}{w_i} \quad \text{if } w_i \neq 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

With, where U ($m \times m$) and V ($n \times n$) are orthogonal and $W(m,n)$ is diagonal with real, non-negative

singular values w_i , we get the pseudoinverse \bar{A} to A as:

$$A = UW\tilde{V} \quad \rightarrow \quad \bar{A} = V\bar{W}\tilde{U}$$

For nonsingular square matrices the pseudoinverse agrees with the common inverse. But the pseudoinverse exists also for singular matrices.

If the rank r of A is smaller than n , $r < n$, then the inverse of A does not exist, and one uses only the first r singular values; W then becomes an $(r \times r)$ matrix and U, V shrink accordingly.

Sources: Rudolf K. Bock, Pseudoinverse. <http://rkb.home.cern.ch/rkb/AN16pp/node220.html>
<http://www.das-gelbe-rechenbuch.de/download/Swz.pdf>

PseudoInverse[m] compute the pseudoinverse of the rectangular matrix m ;
works for numeric and symbolic matrices, real or complex.

Example 1

```
ma = {{1, -2}, {2, -1}, {1, 1}};
mb = {-1, 1, 5};
lx = {x, y};
eq = ma.lx == mb // Thread
{x - 2 y == -1, 2 x - y == 1, x + y == 5}
```

This is an overdetermined system; the coefficient matrix is rectangular, viz. 3×2 . Its rank is 2.

```
MatrixRank[ma]
```

```
2
```

Eqs. (1) and (2) are now illustrated:

```
pa = Transpose[ma].ma
```

```
{{6, -3}, {-3, 6}}
```

```
ba = Inverse[pa].Transpose[ma]
```

```
{{0, 1/3, 1/3}, {-1/3, 0, 1/3}}
```

```
ba.mb
```

```
{2, 2}
```

```
pia = PseudoInverse[ma]
```

```
pia.ma // MatrixForm
```

```
pso = pia.mb
```

```
{{0, 1/3, 1/3}, {-1/3, 0, 1/3}}
```

```
( 1 0 )
( 0 1 )
```

```
{2, 2}
```

Example 2

```
ma = {{1, 2, 3}, {I, 1., -I}} // Transpose; MatrixForm[ma]
```

```
( 1  i )
( 2  1. )
( 3  -i )
```

```
pa = PseudoInverse[ma]; MatrixForm[pa]
```

$$\begin{pmatrix} 0.147059 + 0.0588235 i & 0.117647 + 0.0588235 i & 0.205882 - 0.0588235 i \\ -0.0588235 - 0.470588 i & 0.294118 - 0.117647 i & -0.176471 + 0.235294 i \end{pmatrix}$$

```
pa.ma // Chop // MatrixForm
```

$$\begin{pmatrix} 1. & 0 \\ 0 & 1. \end{pmatrix}$$

```
Inverse[Conjugate[Transpose[ma]].ma].Transpose[ma]
```

$$\{\{0.0294118 - 0.0588235 i, 0.117647 + 0.0588235 i, 0.323529 + 0.0588235 i\}, \{-0.0588235 + 0.352941 i, 0.294118 - 0.117647 i, -0.176471 - 0.588235 i\}\}$$

Example 3

```
Clear[a, b, c, d, m]
```

```
m = {{a, b}, {c, d}}; MatrixForm[m]
```

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

```
Inverse[m] // MatrixForm
```

$$\begin{pmatrix} \frac{d}{-bc+ad} & -\frac{b}{-bc+ad} \\ -\frac{c}{-bc+ad} & \frac{a}{-bc+ad} \end{pmatrix}$$

```
PseudoInverse[m] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \frac{d}{-bc+ad} & \frac{b}{bc-ad} \\ \frac{c}{bc-ad} & \frac{a}{-bc+ad} \end{pmatrix}$$

```
MatrixForm[m = {{1, 2}, {1, 2}}]
```

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

```
Inverse[m]
```

```
Inverse::sing : Matrix {{1, 2}, {1, 2}} is singular. >>
```

```
Inverse[{{1, 2}, {1, 2}}]
```

```
Inverse::sing : Matrix {{1, 2}, {1, 2}} is singular. >>
```

```
MatrixForm[p = PseudoInverse[m]]
```

$$\begin{pmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

This shows that p satisfies the four conditions that uniquely characterize the pseudoinverse p of a matrix m :

```
m.p.m == m
```

```
p.m.p == p
```

```
p.m == Transpose[p.m]
```

```
m.p == Transpose[m.p]
```

```
True
```

```
True
```

```
True
```

```
True
```

```
mm = {{1, 2, 3}, {3, 2, 1}};
```

```
MatrixForm[p = PseudoInverse[mm]]
```

$$\begin{pmatrix} -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{12} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$$

Geometric interpretation of the solution by the pseudoinverse

```
posize = PointSize[0.02];
```

3 independent straight lines

```
c1 = {1, 1};
```

```
c2 = {3, 2};
```

```
c3 = {2, 3};
```

```
cp = Map[Point, {c1, c2, c3}]
```

```
{Point[{1, 1}], Point[{3, 2}], Point[{2, 3}]}
```

```
g1 = x - 2 y == -1
```

```
g2 = 2 x - y == 1
```

```
g3 = x + y == 5
```

```
x - 2 y == -1
```

```
2 x - y == 1
```

```
x + y == 5
```

```
s1 = y /. Solve[g1, y] // Flatten
```

```
s2 = y /. Solve[g2, y] // Flatten
```

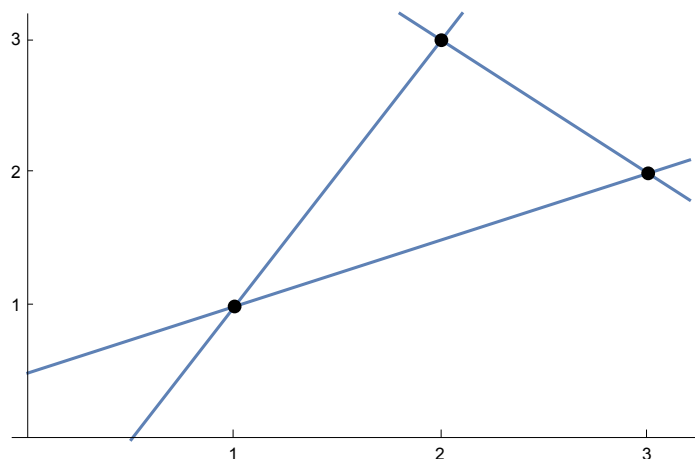
```
s3 = y /. Solve[g3, y] // Flatten
```

$$\left\{ \frac{1+x}{2} \right\}$$

$$\{-1+2x\}$$

$$\{5-x\}$$

```
gr1 = Plot[{s1, s2, s3} // Flatten, {x, 0, 3.2}, PlotRange -> {0, 3.2},
  Ticks -> {Range[3], Range[3]}, Epilog -> Prepend[cp, posize]]
```



```
ma = {{1, -2}, {2, -1}, {1, 1}};
```

```
mb = {-1, 1, 5};
```

```
lx = {x, y};
```

```
eq = ma.lx == mb // Thread
```

```
{x - 2 y == -1, 2 x - y == 1, x + y == 5}
```

```

pia = PseudoInverse[ma]
pia.ma // MatrixForm
pso = pia.mb

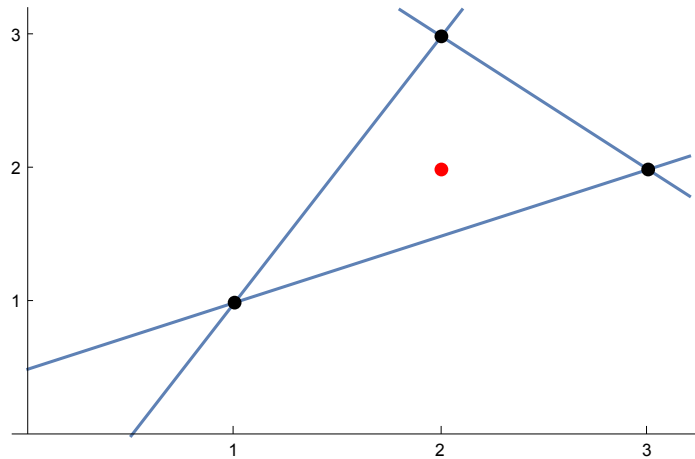
```

$$\left\{ \left\{ 0, \frac{1}{3}, \frac{1}{3} \right\}, \left\{ -\frac{1}{3}, 0, \frac{1}{3} \right\} \right\}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{2, 2\}$$

```
Show[gr1, Graphics[{Hue[0], posize, Point[pso]}]]
```



The "pseudosolution" is the centre of gravity of the triangle.

2 parallel straight lines crossing a third one

```

c1 = {1, 1};
c2 = {3, 2};
c3 = {2, 3};
cp = Map[Point, {c2, c3}];

g1 = x - 2 y == -1;
g2 = x - 2 y == -3.95;
g3 = x + y == 5;

s1 = y /. Solve[g1, y] // Flatten
s2 = y /. Solve[g2, y] // Flatten
s3 = y /. Solve[g3, y] // Flatten

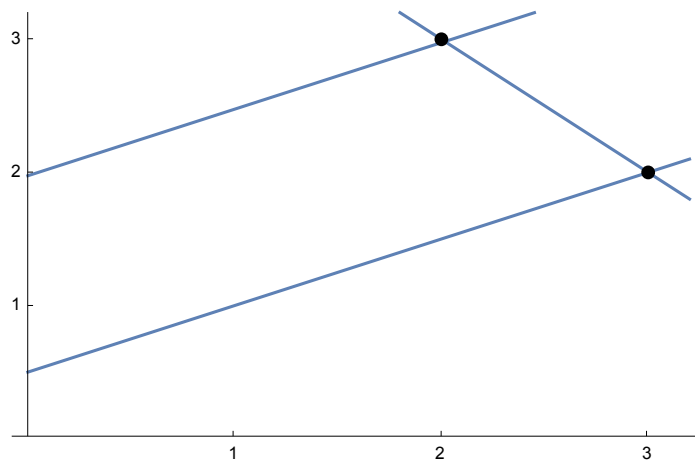
```

$$\left\{ \frac{1+x}{2} \right\}$$

$$\{-0.5(-3.95-x)\}$$

$$\{5-x\}$$

```
gr2 = Plot[{s1, s2, s3} // Flatten, {x, 0, 3.2}, PlotRange -> {0, 3.2},
  Ticks -> {Range[3], Range[3]}, Epilog -> Prepend[cp, posize]]
```



```
ma = {{1, -2}, {1, -2}, {1, 1}}; MatrixForm[ma]
mb = {-1, -3.95, 5};
eq = ma.lx == mb // Thread
```

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \\ 1 & 1 \end{pmatrix}$$

```
{x - 2 y == -1, x - 2 y == -3.95, x + y == 5}
```

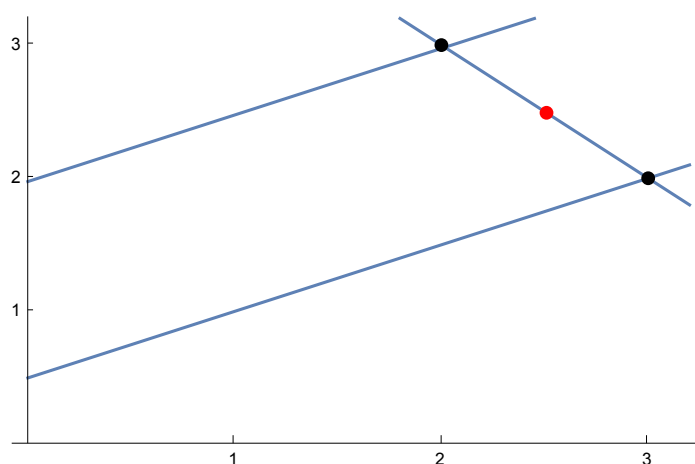
```
pia = PseudoInverse[ma]; MatrixForm[pia]
pia.ma // MatrixForm
pso = pia.mb
```

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
{2.50833, 2.49167}
```

```
Show[gr2, Graphics[{{Hue[0], posize, Point[pso]}]}]
```



The "pseudosolution" is the centre of gravity of the two crossing points..

3 parallel straight lines

```
g1 = x - 2 y == -1;
g2 = x - 2 y == -3.95;
g3 = x - 2 y == -2.1;
```

```

s1 = y /. Solve[g1, y] // Flatten
s2 = y /. Solve[g2, y] // Flatten
s3 = y /. Solve[g3, y] // Flatten

```

$$\left\{ \frac{1+x}{2} \right\}$$

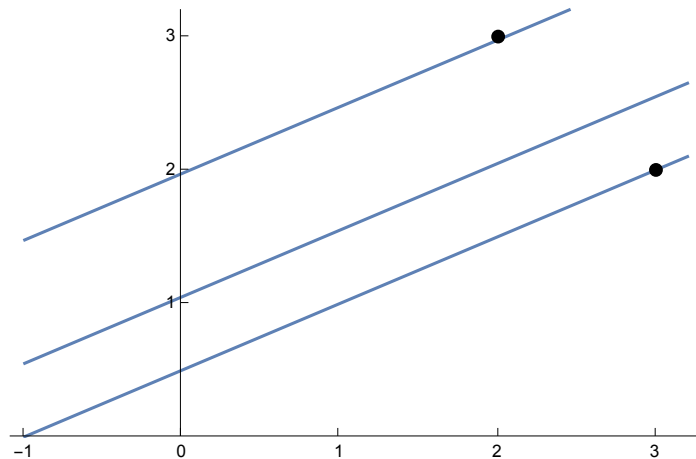
$$\{-0.5(-3.95-x)\}$$

$$\{-0.5(-2.1-x)\}$$

```

gr3 = Plot[{s1, s2, s3} // Flatten, {x, -1, 3.2}, PlotRange -> {0, 3.2},
  Ticks -> {Range[-1, 3], Range[3]}, Epilog -> Prepend[cp, posize]]

```



```

ma = {{1, -2}, {1, -2}, {1, -2}};
mb = {-1, -3.95, -2};
eq = ma.lx == mb // Thread
{x - 2 y == -1, x - 2 y == -3.95, x - 2 y == -2}

```

```

pia = PseudoInverse[ma]
pia.ma // MatrixForm
pso = pia.mb

```

$$\left\{ \left\{ \frac{1}{15}, \frac{1}{15}, \frac{1}{15} \right\}, \left\{ -\frac{2}{15}, -\frac{2}{15}, -\frac{2}{15} \right\} \right\}$$

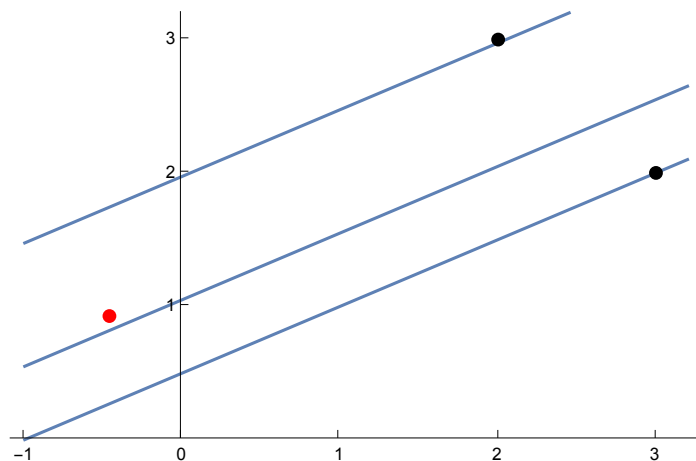
$$\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

$$\{-0.463333, 0.926667\}$$

```

Show[gr3, Graphics[{Hue[0], posize, Point[pso]}]]

```



8.10 Exercises

Only the exercises with a cross (+) are obligatory. If an obligatory exercise comprises several tasks distinguished by a letter, then only one of these tasks is requested.

8.1+ Solve the following systems of linear equations; 3) and 4) are homogeneous systems, the matrix gives the coefficients. Use at least 3 methods when the system is regular, otherwise at least 2 methods.

a) $x + y + z = 1, \quad x + 2y + 3z = 4, \quad x + 3y + 6z = 10, \quad x + 4y + 6z = 19.$

b) $x + y + z = 1, \quad x + 2y + 3z = 4, \quad x + 3y + 6z = 10, \quad x + 4y + 10z = 19.$

c) $\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{pmatrix}.$

8.2+ Replace number 19 in the system 8.1.b) above by an unknown **a**. Without using the results obtained in 8.1.b) find the value of **a** such that the system of 4 equations has a unique solution.

8.3+ Compute the square, the transpose, the trace, the determinant, the inverse, the characteristic polynomial, the latent roots (= eigenvalues) and eigenvectors of the following matrices and transform them into equivalent diagonal matrices.

a) $\begin{pmatrix} 10 & -14 & -10 \\ -14 & 7 & -4 \\ -10 & -4 & 19 \end{pmatrix};$ b) $\begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix};$ c) $\begin{pmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{pmatrix};$ d) $\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 1 \\ \sqrt{3} & 1 & 1 \end{pmatrix};$

e) $\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix};$ f) *) $\begin{pmatrix} 8 & 2 & 3 & 4 \\ 2 & 5 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 9 \end{pmatrix}$

*) It is not reasonable to try to calculate analytic expressions for the eigenvectors of this matrix.

8.4+ Some powers of the matrices given below coincide with the identity matrix; find the corresponding lowest positive power.

a) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ b) $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ c) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ e) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

8.5 a) Write a program, which tests all powers (up to a given limit) of a given matrix **a** to find the lowest positive power of **a** which equals the identity matrix;

b) Write a program, which tests all powers (up to a given limit) of a given matrix **a** to find the lowest positive power of **a** which equals the null matrix (then **a** is nilpotent).

8.6 Write a program, which proves by calculation that a given matrix **a** satisfies its own characteristic equation. (For this task commands described in Chap. 9 may be helpful).

8.7+ A zero divisor of a given matrix **A** is a (square or rectangular) matrix **L** or **R**, which differs from the null matrix, but for which $LA = 0$ (then **L** is a left zero divisor) or $AR = 0$ (then **R** is a right zero divisor). **R**, **L** respectively, consist of the null space of **A**, of the transpose of **A** respectively. Find the left and right zero divisors of the following matrices:

a) $\begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 2 \\ 0 & -8 & -7 \end{pmatrix},$



$$b) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 5 & 7 & 8 \\ 6 & 5 & 4 & 2 \end{pmatrix}.$$



8.8. Determine the shape (determined by the eigenvalues) and the orientation (given by the eigenvectors) of the following conics:

$$1) \quad x^2 + 5y^2 + 3z^2 + 8yz - 8xz = 1;$$

$$2)^* \quad 5x^2 + 13y^2 + 14z^2 - 4xz + 12xy + 12yz = 1.$$

*) It is not reasonable to calculate the eigenvalues or eigenvectors as analytic expressions .

8.9. Compute the Kronecker product of each of the 3 Pauli spin matrices and of the corresponding 2×2 identity matrix e .

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

8.10 A matrix can be diagonalized if it is normal, i.e. if it commutes with its adjoint ($AA^\dagger = A^\dagger A$), or has pairwise different eigenvalues. Write a program which checks this property for a given matrix A . Apply this test to the following matrices:

$$a) \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$



$$b) \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$$



8.12+ Is the following matrix exactly equivalent to a diagonal matrix ?

$$\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

8.13 There is an example due to Wilkinson, which shows that in the following tridiagonal matrix

$$\begin{pmatrix} 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \end{pmatrix}$$

(it is a 21×21 matrix, whose main diagonal is $10, 9, \dots, 1, 0, 1, \dots, 10$, whose parallel diagonals consist of 1's) the elements in the parallel diagonals need not be small if some of the eigenvalues differ by a small amount only. Prepare this matrix by as few commands as possible and compute the eigenvalues.